What Can Routh Table Offer in Addition to Stability?*

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Abstract

Routh stability test is covered in almost all undergraduate control text. It determines the stability or, a little beyond, the number of unstable roots of a polynomial in terms of the signs of certain entries of the Routh table constructed from the coefficients of the polynomial. The use of the Routh table, as far as the common textbooks show, is only limited to this function. In this paper, we will show that the Routh table can actually be used to construct an orthonormal basis in the space of strictly proper rational functions with a common stable denominator. This orthonormal basis can then be used for many other purposes, including the computation of the $\mathcal{H}_2$ norm, the Hankel singular values and singular vectors, model reduction, $\mathcal{H}_\infty$ optimization, etc.

1 Introduction

Recently we have been witnessing a great amount of attention paid to the innovation of undergraduate level control education. Several new textbooks have been published ([5, 17, 13, 8, 4]). The main effort seems to be in incorporating modern and post-modern control theory into the syllabus of a beginners' control course which has been dominated by classical materials for several decades. This effort is not easy and is potentially controversial because of the myth that the modern and post-modern control theory necessitates the use of advanced mathematical knowledge which a typical engineering undergraduate student does not have.

The need to incorporate post-modern control theory into the beginners' course motivates the investigation of the connection between advanced optimal and robust control problems and the classical tools. This paper contains some results from this investigation. We will show that the Routh table can be used to construct an orthonormal basis in the space of strictly proper rational functions with a common stable denominator. This orthonormal basis can then be used to compute the $\mathcal{H}_2$ norm of a transfer function, recovering an algorithm given in [2]. It can also be used to find the Hankel singular values and vectors, hence yielding the solutions to the Hankel approximation and the Nehari problems. This opens the door for a complete and systematic linear optimal and robust control theory using elementary tools not much beyond the well-known Routh stability criterion.

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2 Routh Stability Test and Orthonormal Functions

Consider polynomial

$$a(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_n, \qquad a_0 > 0.$$

Construct the Routh table

s^n	$r_{00} = a_0$	$r_{01} = a_2$	$r_{02} = a_4$	$r_{03} = a_6$	$\cdots$
s^{n-1}	$r_{10} = a_1$	$r_{11} = a_3$	$r_{12} = a_5$	$r_{13} = a_7$	$\cdots$
s^{n-2}	r_{20}	r_{21}	r_{22}	r_{23}	$\cdots$
s^{n-3}	r_{30}	r_{31}	r_{32}	r_{33}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
s^2	$r_{(n-2)0}$	$r_{(n-2)1}$			
s^1	$r_{(n-1)0}$				
s^0	r_{n0}				

Each row starting from the third one is computed from its two preceding rows as

$$r_{ij} = -\frac{1}{r_{(i-1)0}} \begin{vmatrix} r_{(i-2)0} & r_{(i-2)(j+1)} \\ r_{(i-1)0} & r_{(i-1)(j+1)} \end{vmatrix} = \frac{r_{(i-1)0} r_{(i-2)(j+1)} - r_{(i-2)0} r_{(i-1)(j+1)}}{r_{(i-1)0}}.$$

Here i goes from 2 to n and j goes from 0 to $\lfloor \frac{n-i}{2} \rfloor$. When computing the last element of certain row of the Routh table, one may find that the preceding row is one element short of what we need. For example, when we compute r_{n0}, we need $r_{(n-1)1}$ but $r_{(n-1)1}$ is not an element of the Routh table. In this case, we can simply augment the preceding row by a 0 in the end and keep the computation going. Keep in mind that this augmented 0 is not considered as part of the Routh table. Equivalently, whenever $r_{(i-1)(j+1)}$ is missing, simply let $r_{ij} = r_{(i-2)(j+1)}$. For example, r_{n0} can be computed as

$$r_{n0} = -\frac{1}{r_{(n-1)0}} \begin{vmatrix} r_{(n-2)0} & r_{(n-2)1} \\ r_{(n-1)0} & \mathbf{0} \end{vmatrix} = r_{(n-2)1}.$$

Theorem 1 (Routh Stability Criterion) *The following statements are equivalent:*

1. *$p(s)$ is stable.*
2. *All elements of the Routh table are positive, i.e., $r_{ij} > 0$, $i = 0, 1, \ldots, n$, $j = 0, 1, \ldots, \lfloor \frac{n-i}{2} \rfloor$.*
3. *All elements in the first column of the Routh table are positive, i.e., $r_{i0} > 0$, $i = 0, 1, \ldots, n$.*

In general, the Routh table cannot be completely constructed when an element in the first column is zero. In this case, as far as the stability of the polynomial is concerned, there is no need to complete the rest of the table since we already know from the Routh criterion that the polynomial is unstable.

The proof given by Routh is quite involved and is usually omitted in feedback control textbooks. There have been continuous efforts in finding simpler proofs. It appears that the proof given in [2, Section 5.3] uses the most elementary arguments and is the most easily understandable. Interestingly, this proof was rediscovered at least a couple of times by [15] and [6].

Let us now fix a stable polynomial

$$a(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n, \quad a_0 > 0.$$

Consider the set of rational functions

$$\mathcal{S}_{a(s)} = \left\{ \frac{b(s)}{a(s)} : b(s) = b_1 s^{n-1} + \cdots + b_{n-1} s + b_n, b_i \in \mathbb{R}, i = 1, \ldots, n \right\}.$$

This set is clearly a subspace of $\mathcal{H}_2$. We will see that an orthonormal basis of this subspace is very useful in various purposes.

Let us construct the Routh table of $a(s)$. Since $a(s)$ is stable, the Routh table can always be constructed to the end and all $r_{i0}, i = 0, 1, \ldots, n$, are positive. For each row (except the first one) of the Routh table, define a polynomial

$$\begin{aligned} r_1(s) &= r_{10} s^{n-1} + r_{11} s^{n-3} + \cdots \\ r_2(s) &= r_{20} s^{n-2} + r_{21} s^{n-4} + \cdots \\ &\vdots \\ r_{n-1}(s) &= r_{(n-1)0} s \\ r_n(s) &= r_{n0}. \end{aligned}$$

Also define

$$\alpha_i = \frac{r_{(i-1)0}}{r_{i0}}, \quad i = 1, 2, \ldots, n.$$

Theorem 2 *The functions*

$$B_i(s) = \sqrt{2\alpha_i} \frac{r_i(s)}{a(s)}, \quad i = 1, 2, \ldots, n,$$

form an orthonormal basis of $\mathcal{S}_{a(s)}$.

Proof: There are a number of ways to prove this theorem. The proof presented here reveals some interesting connections to the state space system theory. It is known [9] that if $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is a balanced realization of an inner transfer function. Then the elements of $C(sI - A)^{-1}$ are orthonormal. In the following we will prove that the realization

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{ccccc|c} -\frac{r_{10}}{r_{00}} & \sqrt{\frac{r_{20}}{r_{00}}} & & & & -\sqrt{\frac{2r_{10}}{r_{00}}} \\ -\sqrt{\frac{r_{20}}{r_{00}}} & 0 & \ddots & & & 0 \\ & \ddots & \ddots & \ddots & & \vdots \\ & & \ddots & 0 & \sqrt{\frac{r_{n0}}{r_{(n-2)0}}} & 0 \\ & & & -\sqrt{\frac{r_{n0}}{r_{(n-2)0}}} & 0 & 0 \\ \hline \sqrt{\frac{2r_{10}}{r_{00}}} & 0 & \cdots & \cdots & 0 & 1 \end{array}\right], \quad (1)$$

where the A matrix is named the Routh canonical form in [3], is a balanced realization of the inner transfer function

$$U(s) = (-1)^n \frac{a(-s)}{a(s)} = 1 - 2\frac{r_1(s)}{a(s)},$$

and exactly

$$C(sI - A)^{-1} = \begin{bmatrix} B_1(s) & \cdots & B_n(s) \end{bmatrix}. \tag{2}$$

We first prove equality (2). Denote

$$C(sI - A)^{-1} = \begin{bmatrix} \frac{c_1(s)}{c(s)} & \cdots & \frac{c_n(s)}{c(s)} \end{bmatrix}.$$

Then the identity $C(sI - A)^{-1}(sI - A) = C$ implies

$$\left(s + \frac{r_{10}}{r_{00}}\right) c_1(s) + \sqrt{\frac{r_{20}}{r_{00}}} c_2(s) = \sqrt{\frac{2r_{10}}{r_{00}}} c(s) \tag{3}$$

$$-\sqrt{\frac{r_{(i+1)0}}{r_{(i-1)0}}} c_i(s) + s c_{i+1}(s) + \sqrt{\frac{r_{(i+2)0}}{r_{i0}}} c_{i+2}(s) = 0, \qquad i = 1, 2, \ldots, n-2, \tag{4}$$

$$-\sqrt{\frac{r_{n0}}{r_{(n-2)0}}} c_{n-1}(s) + s c_n(s) = 0. \tag{5}$$

This shows that $c_i(s)$ satisfies backward difference equation

$$c_i(s) = \sqrt{\frac{r_{(i-1)0} r_{(i+2)0}}{r_{i0} r_{(+1)0}}} c_{i+2}(s) + \sqrt{\frac{r_{(i-1)0}}{r_{(i+1)0}}} s c_{i+1}(s)$$

with terminal conditions

$$c_n(s) = \sqrt{\frac{r_{(n-1)0} r_{n0}}{r_{00} r_{10}}}, \quad c_{n-1}(s) = \sqrt{\frac{r_{(n-2)0} r_{(n-1)0}}{r_{00} r_{10}}} s.$$

On the other hand, the construction of Routh table implies

$$r_i(s) = r_{i+2}(s) + \frac{r_{i0}}{r_{(i+1)0}} s r_{i+1}(s), \quad i = 0, 1, \ldots, n-2,$$

with terminal conditions

$$r_n(s) = r_n, \quad r_{n-1}(s) = r_{n-1} s.$$

Here $r_0(s)$ is a polynomial defined from the first row of the Routh table

$$r_0(s) = a_0 s^n + a_2 s^{n-2} + a_4 s^{n-4} + \cdots,$$

and we have $r_0(s) + r_1(s) = a(s)$. If we define

$$b_i(s) = \sqrt{\frac{2r_{(i-1)0}}{r_{i0}}} r_i(s), \quad i = 1, 2, \ldots, n, \tag{6}$$

then the polynomials $b_i(s), i = 1, 2, \ldots, n-2$, satisfy difference equation

$$b_i(s) = \sqrt{\frac{r_{(i-1)0} r_{(i+2)0}}{r_{i0} r_{(i+1)0}}} b_{i+2}(s) + \sqrt{\frac{r_{(i-1)0}}{r_{(i+1)0}}} s b_{i+1}(s).$$

with terminal condition

$$b_n(s) = \sqrt{2r_{(n-1)0}r_{n0}}, \quad b_{n-1}(s) = \sqrt{2r_{(n-2)0}r_{(n-1)0}}s.$$

We see that the polynomials $b_i(s)$ and $c_i(s)$ satisfy the same linear difference equation with terminal conditions differ only by a scalar. It then follows that $b_i(s)$ and $c_i(s)$ differ by the same scalar for all i, i.e.,

$$c_i(s) = \frac{b_i(s)}{\sqrt{2r_{00}r_{10}}}, \quad i = 1, 2, \ldots, n. \tag{7}$$

From (3), (6) and (7), we see that

$$\begin{aligned} c(s) &= \sqrt{\frac{r_{00}}{2r_{10}}}\left(s + \frac{r_{10}}{r_{00}}\right)c_1(s) + \sqrt{\frac{r_{20}}{2r_{10}}}c_2(s) \\ &= \frac{1}{2r_{10}}\left(s + \frac{r_{10}}{r_{00}}\right)b_1(s) + \sqrt{\frac{r_{20}}{r_{00}}}\frac{1}{2r_{10}}b_2(s) \\ &= \frac{1}{\sqrt{2r_{00}r_{10}}}\left[s\frac{r_{00}}{r_{10}}r_1(s) + r_1(s) + r_2(s)\right] \\ &= \frac{1}{\sqrt{2r_{00}r_{10}}}[r_0(s) + r_1(s)] \\ &= \frac{a(s)}{\sqrt{2r_{00}r_{10}}}. \end{aligned}$$

Therefore

$$C(sI - A)^{-1} = \begin{bmatrix} \frac{c_1(s)}{c(s)} & \cdots & \frac{c_n(s)}{c(s)} \end{bmatrix} = \begin{bmatrix} \frac{b_1(s)}{a(s)} & \cdots & \frac{b_n(s)}{a(s)} \end{bmatrix} = \begin{bmatrix} B_1(s) & \cdots & B_n(s) \end{bmatrix}.$$

Next, we prove the realization has transfer function $U(s)$. This follows immediately from

$$D + C(sI - A)^{-1}B = 1 - \sqrt{\frac{2r_{10}}{r_{00}}}\frac{b_1(s)}{a(s)} = 1 - 2\frac{r_1(s)}{a(s)}.$$

Finally, it is obvious that the realization (1) is balanced since its controllability and observability gramians are both equal to the identity matrix. □

3 Computation of the RMS value

Given a strictly proper stable signal or system

$$G(s) = \frac{b(s)}{a(s)} = \frac{b_1 s^{n-1} + \cdots + b_n}{a_0 s^n + a_1 s^{n-1} + \cdots + a_n}, \quad a_0 > 0.$$

Clearly $G(s) \in \mathcal{S}_{a(s)}$.

If we expand $b(s)$ as

$$b(s) = \beta_1 r_1(s) + \beta_2 r_2(s) + \cdots + \beta_n r_n(s), \tag{8}$$

then

$$G(s) = \frac{\beta_1}{\sqrt{2\alpha_1}}B_1(s) + \frac{\beta_2}{\sqrt{2\alpha_2}}B_2(s) + \cdots + \frac{\beta_n}{\sqrt{2\alpha_n}}B_n(s).$$

Consequently

$$\|G(s)\|_2^2 = \frac{\beta_1^2}{2\alpha_1} + \frac{\beta_2^2}{2\alpha_2} + \cdots + \frac{\beta_n^2}{2\alpha_n}.$$

It seems that finding all β_i requires solving a set of linear equations obtained by comparing the coefficients in (8). Actually, these equations have special structure which leads to a tabular solution. Construct the augmented Routh table:

s^n	$r_{00} = a_0$	$r_{01} = a_2$	$\cdots$	$q_{00} = b_1$	$q_{01} = b_3$	$\cdots$			
s^{n-1}	$r_{10} = a_1$	$r_{11} = a_3$	$\cdots$	r_{10}	r_{11}	$\cdots$	$q_{10} = b_2$	$q_{11} = b_4$	$\cdots$
s^{n-2}	r_{20}	r_{21}	$\cdots$	q_{20}	q_{21}	$\cdots$	r_{20}	r_{21}	$\cdots$
s^{n-3}	r_{30}	r_{31}	$\cdots$	r_{30}	r_{31}	$\cdots$	q_{30}	q_{31}	$\cdots$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
s	$r_{(n-1)0}$								
s^0	r_{n0}								

The augmented Routh table is formed by adding two blocks to the right of the usual Routh table. The first added block (the middle block of the augmented Routh table) is constructed in the following way: the first row is directly from the coefficients $b_1, b_3, b_5, \ldots,$ of $b(s)$; the second, forth, sixth, ..., rows are copied from the corresponding rows of the Routh table; the third, fifth, seventh, ..., rows are obtained from their preceding two rows in exactly the same way as the rows of the Routh table:

$$q_{ij} = -\frac{1}{r_{(i-1)0}} \begin{vmatrix} q_{(i-2)0} & q_{(i-2)(j+1)} \\ r_{(i-1)0} & r_{(i-1)(j+1)} \end{vmatrix}. \tag{9}$$

The second added block (the right block of the augmented Routh table) is constructed in the following way: the first row is irrelevant, the second row is directly from the coefficients $b_2, b_4, b_6, \ldots,$ of $b(s)$, the third, fifth, seventh, ..., rows are copied from the corresponding rows of the Routh table, the forth, sixth, eighth, ..., rows are obtained from their preceding two rows in exactly the same way as the rows of the Routh table using formula (9).

In summary, the following algorithm gives the 2-norm of a stable strictly proper transfer function.

Algorithm 1

Step 1 *Compute the augmented Routh table of* $G(s)$.

Step 2 *Set* $\alpha_i = \frac{r_{(i-1)0}}{r_{i0}}$ *and* $\beta_i = \frac{q_{(i-1)0}}{r_{i0}}$.

Step 3 $\|G(s)\|_2 = \sqrt{\frac{\beta_1^2}{2\alpha_1} + \frac{\beta_2^2}{2\alpha_2} + \cdots + \frac{\beta_n^2}{2\alpha_n}}$.

The effort to find a simple method to compute the RMS value of a transfer function started in the late 40's by a group in MIT. The initial effort ended up with formulas for transfer functions up to 7th order, reported in [11]. Another team effort was carried out in the 50's by another group in MIT. This effort, documented in [16], led to an algorithm based on matrix equation for arbitrarily high order transfer functions and corrections to two formulas in [11]. Algorithm 1 in this section is not new and first appeared in [2, Section 5.3]. What is new here is the observation that this algorithm directly follows from the availability of an orthonormal basis of $\mathcal{S}_{a(s)}$.

Example 1
Consider

$$G(s) = \frac{b(s)}{a(s)} = \frac{s^3 + 2s^2 + 5s + 6}{s^4 + s^3 + 3s^2 + 2s + 1}.$$

The augmented Routh table of $G(s)$ is

s^4	$a_0 = 1$	$a_2 = 3$	$a_4 = 1$	$b_1 = 1$	$b_3 = 5$		
s^3	$a_1 = 1$	$a_3 = 2$		1	2	$b_2 = 2$	$b_4 = 6$
s^2	1	1		3		1	1
s^1	1			1		4	
s^0	1					1	

Therefore

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$$

and

$$\beta_1 = 1, \quad \beta_2 = 2, \quad \beta_3 = 3, \quad \beta_4 = 4.$$

Hence

$$\|G(s)\|_2^2 = \frac{1^2}{2} + \frac{2^2}{2} + \frac{3^2}{2} + \frac{4^2}{2} = 15.$$

4 Hankel Singular Values and Vectors

Given a proper stable transfer function:

$$G(s) = \frac{b(s)}{a(s)},$$

take a function $\frac{x(s)}{a(s)}$ in $\mathcal{S}_{a(s)}$. Then

$$G(s)\frac{x(-s)}{a(-s)} = \frac{b(s)x(-s)}{a(s)a(-s)}$$

is a strictly proper rational function with poles at the roots of $a(s)$ and their mirror images with respect to the imaginary axis. This rational function can be uniquely decomposed into

$$\frac{b(s)x(-s)}{a(s)a(-s)} = \frac{y(s)}{a(s)} + \frac{z(s)}{a(-s)}$$

where both terms on the right hand side are strictly proper. Throw away the unstable term $\frac{z(s)}{a(-s)}$. Then we are left with the stable term $\frac{y(s)}{a(s)}$ which belongs to $\mathcal{S}_{a(s)}$. This process defines a map from $\mathcal{S}_{a(s)}$ to $\mathcal{S}_{a(s)}$:

$$\frac{x(s)}{a(s)} \mapsto \frac{y(s)}{a(s)}.$$

This map consists of three actions. The first is reversion; this action simply replaces the variable s in $\frac{x(s)}{a(s)}$ by $-s$, resulting in $\frac{x(-s)}{a(-s)}$. The second is multiplication; this action multiplies the result of the first action by $G(s)$. The third action is projection; it keeps the stable part of the result of the second action and throw away the unstable part. Since all these actions are linear operations. The map is clearly a linear operator on $\mathcal{S}_{a(s)}$. We call it the Hankel operator with symbol $G(s)$, denoted by $H_{G(s)}$.

Readers who are familiar with the usual definition of the Hankel operator as a map between infinite dimensional spaces $\mathcal{H}_2^\perp$ to $\mathcal{H}_2$ may wonder how to reconcile the two definitions. Actually, the finite dimensional Hankel operator defined above is essentially the nontrivial part of the usual infinite dimensional Hankel operator after being compressed to an operator from the orthogonal complement of its kernel to its range.

A proper $G(s) = \frac{b(s)}{a(s)}$ can in general be decomposed as the sum of a constant term and a strictly proper term

$$G(s) = d + \frac{c(s)}{a(s)}$$

where $d = G(\infty)$ and $c(s) = b(s) - G(\infty)a(s)$. For a function $\frac{\alpha(s)}{a(s)} \in \mathcal{S}_{a(s)}$, let

$$\frac{c(s)x(-s)}{a(s)a(-s)} = \frac{y(s)}{a(s)} + \frac{z(s)}{a(-s)},$$

i.e.,

$$H_{\frac{c(s)}{a(s)}} \frac{x(s)}{a(s)} = \frac{y(s)}{a(s)}.$$

Then

$$G(s)\frac{x(-s)}{a(-s)} = \frac{y(s)}{a(s)} + \frac{z(s) - dx(-s)}{a(-s)},$$

i.e,

$$H_{G(s)} \frac{x(s)}{a(s)} = \frac{y(s)}{a(s)} = H_{\frac{c(s)}{a(s)}} \frac{x(s)}{a(s)}.$$

This shows that the Hankel operator does not depend on d, the constant term in $G(s)$. In other words, the Hankel operator with symbol $G(s)$ is the same as the Hankel operator with symbol $\frac{c(s)}{a(s)}$, which is the strictly proper part of $G(s)$. Hence in the computation related to a Hankel operator, one can disregard the constant part of the symbol.

The Hankel operator can be represented by a matrix if a basis in $\mathcal{S}_{a(s)}$ is chosen. Naturally we can use the orthonormal basis

$$\{B_1(s), B_2(s), \ldots, B_n(s)\}$$

defined from the Routh table of $a(s)$ as in Theorem 2. The matrix representation under this basis is called the Routh-Hankel matrix and is denoted by $R_{G(s)}$. The singular values of $R_{G(s)}$ are called the Hankel singular values of $G(s)$ and are denoted by $\sigma_1(G(s)), \sigma_2(G(s)), \ldots, \sigma_n(G(s))$. Here we assume that the singular values are ordered in a non-increasing way, i.e., we assume that $\sigma_1(G(s)) \geq \sigma_2(G(s)) \geq \cdots \geq \sigma_n(G(s))$. In particular, the largest Hankel singular value $\sigma_1(G(s))$ is called the Hankel norm of $G(s)$ and is denoted by $\|G(s)\|_H$. Let (u_i, v_i) be a pair of left and right singular vectors of $R_{G(s)}$ corresponding to singular value $\sigma_i(G(s))$ and let

$$U_i(s) = \begin{bmatrix} B_1(s) & B_2(s) & \cdots & B_n(s) \end{bmatrix} u_i$$

and

$$V_i(s) = \begin{bmatrix} B_1(s) & B_2(s) & \cdots & B_n(s) \end{bmatrix} v_i.$$

Then $(U_i(s), V_i(s))$ is called a Schmidt pair of $H_{G(s)}$ corresponding to $\sigma_i(G(s))$.

It seems that the Hankel singular values and the corresponding Schmidt pairs defined here depend on the choice of a basis in $\mathcal{S}_{a(s)}$. Actually this is not the case. As long the basis is an orthonormal one, we will end up with the same singular values and Schmidt pairs.

If we are interested in computing the Hankel singular values and Schmidt pairs of $H_{G(s)}$, then the key is to find $R_{G(s)}$ from $G(s) = \frac{b(s)}{a(s)}$. Again, we call the Routh table into action.

Theorem 3 *Construct the Routh table of $a(s)$. Let*

$$A = \begin{bmatrix} -\dfrac{r_{10}}{r_{00}} & \sqrt{\dfrac{r_{20}}{r_{00}}} & & & \\ -\sqrt{\dfrac{r_{20}}{r_{00}}} & 0 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 0 & \sqrt{\dfrac{r_{n0}}{r_{(n-2)0}}} \\ & & & -\sqrt{\dfrac{r_{n0}}{r_{(n-2)0}}} & 0 \end{bmatrix}.$$

Also let $r_1(s)$ be the polynomial defined from the second row of the Routh table:

$$r_1(s) = r_{10}s^{n-1} + r_{11}s^{n-3} + \cdots,$$

and let $c(s)$ be the denominator of the strictly proper part of $G(s)$:

$$c(s) = b(s) - G(\infty)a(s).$$

Then

$$R_{G(s)} = a(-A)^{-1}b(A) \begin{bmatrix} (-1)^{n-1} & & & \\ & \ddots & & \\ & & -1 & \\ & & & 1 \end{bmatrix} \tag{10}$$

$$= [2r_1(-A)]^{-1}c(A) \begin{bmatrix} (-1)^{n-1} & & & \\ & \ddots & & \\ & & -1 & \\ & & & 1 \end{bmatrix}. \tag{11}$$

Proof It is well-known that a rational function $\frac{v(s)}{u(s)}$ can be written as

$$\frac{v(s)}{u(s)} = q(s) + \frac{r(s)}{u(s)}.$$

Here $q(s)$ is called the quotient and $r(s)$ is called the remainder of $v(s)$ divided by $u(s)$. Define a linear map T on rational functions by

$$T\frac{v(s)}{u(s)} = \frac{r(s)}{u(s)}.$$

In other words, the action of T is to keep the strictly proper part of the rational function and throw away the polynomial part. It is easy to see that T is a projection. Define linear transformation $D_{a(s)} : \mathcal{S}_{a(s)} \to \mathcal{S}_{a(s)}$ by

$$D_{a(s)} \frac{\alpha(s)}{a(s)} = T \frac{s\alpha(s)}{a(s)}.$$

Clearly,

$$D_{a(s)}^i \frac{\alpha(s)}{a(s)} = T \frac{s^i \alpha(s)}{a(s)}$$

and

$$a(D_{a(s)}) = 0.$$

The linear transformation $D_{a(s)}$ will be called a differential operator. What is its matrix representation under the orthonormal basis $\{B_i(s) : i = 1, 2, \ldots, n\}$ of $\mathcal{S}_{a(s)}$? Recall the definition of $r_i(s), i = 1, 2, \ldots, n$, in Section 2 and also augment the list by

$$r_0(s) = r_{00}s^n + r_{01}s^{n-2}$$

formed by the first row of the Routh table. The construction of the Routh table means

$$sr_i(s) = \frac{r_{i0}}{r_{(i-1)0}}[r_{i-1}(s) - r_{i+1}(s)], \quad i = 1, 2, \ldots, n-1.$$

Therefore,

$$\begin{aligned} D_{a(s)}B_1(s) &= T\sqrt{\frac{2r_{00}}{r_{10}}}\frac{sr_1(s)}{a(s)} = \sqrt{\frac{2r_{00}}{r_{10}}}\left[\frac{sr_1(s)}{a(s)} - \frac{r_{10}}{r_{00}}\right] \\ &= \sqrt{\frac{2r_{00}}{r_{10}}}\frac{r_{10}}{r_{00}}\frac{[r_0(s) - r_2(s)] - [r_0(s) + r_1(s)]}{a(s)} = -\sqrt{\frac{2r_{10}}{r_{00}}}\frac{r_1(s) + r_2(s)}{a(s)} \\ &= -\frac{r_{10}}{r_{00}}B_1(s) - \sqrt{\frac{r_{20}}{r_{00}}}B_2(s). \end{aligned}$$

For $i = 2, \ldots, n-1$,

$$\begin{aligned} D_{a(s)}B_i(s) &= T\sqrt{\frac{2r_{(i-1)0}}{r_{i0}}}\frac{sr_i(s)}{a(s)} \\ &= \sqrt{\frac{2r_{(i-1)0}}{r_{i0}}}\frac{r_{i0}}{r_{(i-1)0}}\frac{r_{i-1}(s) - r_{i+1}(s)}{a(s)} \\ &= \sqrt{\frac{r_{i0}}{r_{(i-2)0}}}B_{i-1}(s) - \sqrt{\frac{r_{(i+1)0}}{r_{(i-1)0}}}B_{i+1}(s). \end{aligned}$$

Finally

$$D_{a(s)}B_n(s) = T\sqrt{\frac{2r_{(n-1)0}}{r_{n0}}}\frac{sr_n(s)}{a(s)} = \sqrt{\frac{2r_{(n-1)0}}{r_{n0}}}\frac{r_{n0}s}{a(s)} = \sqrt{\frac{r_{n0}}{r_{(n-2)0}}}B_{n-1}(s).$$

In the matrix form,

$$D_{a(s)}\begin{bmatrix}B_1(s) \cdots B_n(s)\end{bmatrix} = \begin{bmatrix}B_1(s) \cdots B_n(s)\end{bmatrix} A.$$

This shows that the matrix representation of $D_{a(s)}$ is exactly A.

Now let us look at the basic equation in defining the Hankel operator

$$\frac{b(s)x(-s)}{a(s)a(-s)} = \frac{y(s)}{a(s)} + \frac{z(s)}{a(-s)}.$$

This can be rewritten as

$$b(s)\frac{x(-s)}{a(s)} = a(-s)\frac{y(s)}{a(s)} + a(s)\frac{z(s)}{a(s)}.$$

Applying the projection T to both sides and using the fact that $a(D_{a(s)}) = 0$, we obtain

$$b(D_{a(s)})\frac{x(-s)}{a(s)} = a(-D_{a(s)})\frac{y(s)}{a(s)}.$$

Under the basis used, we have

$$b(A)\begin{bmatrix} (-1)^{n-1} & & & \\ & \ddots & & \\ & & -1 & \\ & & & 1 \end{bmatrix} = a(-A)R_{G(s)}.$$

Hence

$$R_{G(s)} = a(-A)^{-1}b(A)\begin{bmatrix} (-1)^{n-1} & & & \\ & \ddots & & \\ & & -1 & \\ & & & 1 \end{bmatrix}.$$

Finally, it follows from $b(s) = da(s) + c(s)$ that $b(A) = c(A)$ and it follows from $a(-s) = r_0(-s) + r_1(-s) = \pm a(s) + 2r_1(-s)$ that $a(-A) = 2r_1(-A)$. □

Formula (11) is a bit simpler than formula (10) since $r_1(s)$ and $c(s)$ has lower degree and fewer terms than $a(s)$ and $b(s)$ respectively.

Example 2

For

$$G(s) = \frac{2\sqrt{2}s + 4}{s^2 + \sqrt{2}s + 1},$$

the Routh table gives

$$A = \begin{bmatrix} -\sqrt{2} & 1 \\ -1 & 0 \end{bmatrix}$$

and

$$r_1(s) = \sqrt{2}s, \quad r_2(s) = 1.$$

Both (10) and (11) give

$$R_{G(s)} = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}.$$

The singular values of $R_{G(s)}$ are

$$\sigma_1(G(s)) = \sqrt{2} + 1, \quad \sigma_2(G(s)) = \sqrt{2} - 1,$$

and the corresponding singular vectors are

$$\begin{bmatrix} u_1 & u_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}, \quad \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} \end{bmatrix}.$$

Hence the corresponding Schmidt pairs are

$$\begin{aligned}
(U_1(s), V_1(s)) &= \left(\frac{1}{2}\begin{bmatrix} \frac{\sqrt{2\sqrt{2}}s}{a(s)} & \frac{\sqrt{2\sqrt{2}}}{a(s)} \end{bmatrix}\begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}, \frac{1}{2}\begin{bmatrix} \frac{\sqrt{2\sqrt{2}}s}{a(s)} & \frac{\sqrt{2\sqrt{2}}}{a(s)} \end{bmatrix}\begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}\right) \\
&= \left(\frac{\sqrt[4]{2}(s+1)}{s^2+\sqrt{2}s+1}, \frac{\sqrt[4]{2}(s+1)}{s^2+\sqrt{2}s+1}\right), \\
(U_2(s), V_2(s)) &= \left(\frac{1}{2}\begin{bmatrix} \frac{\sqrt{2\sqrt{2}}s}{a(s)} & \frac{\sqrt{2\sqrt{2}}}{a(s)} \end{bmatrix}\begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \end{bmatrix}, \frac{1}{2}\begin{bmatrix} \frac{\sqrt{2\sqrt{2}}s}{a(s)} & \frac{\sqrt{2\sqrt{2}}}{a(s)} \end{bmatrix}\begin{bmatrix} \sqrt{2} \\ -\sqrt{2} \end{bmatrix}\right) \\
&= \left(\frac{\sqrt[4]{2}(-s+1)}{s^2+\sqrt{2}s+1}, \frac{\sqrt[4]{2}(s-1)}{s^2+\sqrt{2}s+1}\right).
\end{aligned}$$

It is seen that in this example that $R_{G(s)}$ is a symmetric matrix and the left singular vectors are either the same or the negative of the right singular vectors. This is by no means an accident. It can be shown that $R_{G(s)}$ is always a symmetric matrix. This implies that its singular values are the absolute values of its eigenvalues and its left and right singular vectors are essentially the eigenvectors. This fact may offer some simplification in the computation.

5 Hankel Approximation and the Nehari Problem

The problem of Hankel approximation is to find a lower order system to approximate a high order system so that the Hankel norm of the error is minimized. Precisely, if we are given a stable strictly proper transfer function

$$G(s) = \frac{b(s)}{a(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{a_0 s^n + a_1 s^{n-1} + \cdots + a_n}, \quad a_0 > 0,$$

we wish to find

$$\min_{\text{order}\tilde{G}(s) \leq r} \|G(s) - \tilde{G}(s)\|_H$$

and a minimizing $\tilde{G}(s)$. Here we assume that $r < n$.

Before going into the solution of the Hankel approximation problem, we first notice that since the Hankel norm is independent of the constant term, if $\tilde{G}(s)$ is a minimizing solution then $\tilde{G}(s) + c$ is also a minimizing solution for any $c \in \mathbb{R}$. This freedom on the constant term can be used for other considerations.

We need to introduce a machinery. Let $F(s) = \frac{g(s)}{f(s)}$ be an arbitrary rational function. It is well known that $F(s)$ can be decomposed in a unique way as the sum of two rational functions

$$\frac{g(s)}{f(s)} = \frac{\alpha(s)}{f_s(s)} + \frac{\beta(s)}{f_u(s)}$$

where $f_s(s)$ is strictly proper stable and $f_u(s)$ is antitable. The action of the projection operator P is simply to take the stable strictly proper part, i.e., P defined as

$$P\frac{g(s)}{f(s)} = \frac{\alpha(s)}{f_s(s)}.$$

Theorem 4 *Let $(U_{r+1}(s), V_{r+1}(s))$ be the Schmidt pair of $G(s)$ corresponding to $(r+1)$-st Hankel singular value $\sigma_{r+1}(G(s))$. Then*

$$\min_{\text{order}\tilde{G}(s)\leq r} \|G(s) - \tilde{G}(s)\|_H = \sigma_{r+1}(G(s)).$$

and all minimizing $\tilde{G}(s)$ are given by

$$\tilde{G}(s) = G(s) - P\left[\sigma_{r+1}(G(s))\frac{U_{r+1}(s)}{V_{r+1}(-s)}\right] + c$$

for $c \in \mathbb{R}$.

Example 3
We wish to find the 1st order Hankel approximation $\tilde{G}(s)$ of

$$G(s) = \frac{2\sqrt{2}s + 4}{s^2 + \sqrt{2}s + 1}.$$

Following Example 2,

$$\min_{\text{order}\tilde{G}(s)\leq 1} \|G(s) - \tilde{G}(s)\|_H = \sigma_2(G(s)) = \sqrt{2} - 1$$

and all best approximations are given by

$$\tilde{G}(s) = \frac{2\sqrt{2}s + 4}{s^2 + \sqrt{2}s + 1} - P\left[(\sqrt{2}-1)\frac{\sqrt[4]{2}(-s+1)(s^2 - \sqrt{2}s + 1)}{\sqrt[4]{2}(-s-1)(s^2 + \sqrt{2}s + 1)}\right] + c = \frac{2 + 2\sqrt{2}}{s+1} + c$$

for $c \in \mathbb{R}$.

The Nehari problem is as follows: Given stable strictly proper system $G(s) = \frac{b(s)}{a(s)}$, find

$$\min_{Q(s)\in\mathcal{H}_\infty} \|G(-s) - Q(s)\|_\infty$$

and a minimizing $Q(s) \in \mathcal{H}_\infty$.

Theorem 5

$$\min_{Q(s)\in\mathcal{H}_\infty} \|G(-s) - Q(s)\|_\infty = \|G(s)\|_H$$

and if $(U_1(s), V_1(s))$ is a Schmidt pair of the $H_{G(s)}$ corresponding to the largest Hankel singular value $\sigma_1(G(s))$, then the unique optimal $Q(s)$ is given by

$$Q(s) = G(-s) - \sigma_1(G(s))\frac{U_1(-s)}{V_1(s)}.$$

Example 4
For

$$G(s) = \frac{b(s)}{a(s)} = \frac{2\sqrt{2}s + 4}{s^2 + \sqrt{2}s + 1},$$

we wish to find $Q(s) \in \mathcal{H}_\infty$ to minimize

$$\|G(-s) - Q(s)\|_\infty.$$

It follows from Theorem 5 and Example 2 that

$$\min_{Q(s)\in\mathcal{H}_\infty} \|G(-s)-Q(s)\|_\infty = 1+\sqrt{2}$$

and the optimal $Q(s)$ is given by

$$Q(s) = \frac{-2\sqrt{2}s+4}{s^2-\sqrt{2}s+1} - (1+\sqrt{2})\frac{\sqrt[4]{2}(-s+1)(s^2+\sqrt{2}s+1)}{\sqrt[4]{2}(s+1)(s^2-\sqrt{2}s+1)} = \frac{(1+\sqrt{2})s+3\sqrt{2}-1}{s+1}.$$

The theorems in this section are well-known and were commonly credited to [1], see also the excellent exposition [18]. The novelty here is that the required Schmidt pairs can be computed by means of the Routh table. Routh table was used for model reduction before [10] but the method there has nothing to do with the Hankel approximation.

6 Concluding Remarks

The popular method of computing the $\mathcal{H}_2$ norm, the Hankel singular values and the Schmidt pairs of the Hankel operator is through the state space model and Lyapunov equations, see [19]. The alternative method given in this paper, growing out of the classical Routh table, has apparent advantages, at least for SISO systems. It is conceptually simpler, numerically less complex, and mathematically less sophisticated. It well serves the original motivation for its development: the accessibility for undergraduate students and practicing engineers with minimal mathematical background. The method can also be extended to MIMO systems in an obvious way. Even in the MIMO case, this state space free method has its distinct merit compared to the state space method.

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