General Multirate Building Structures with Application to Nonuniform Filter Banks

Tongwen Chen, *Senior Member, IEEE,* Li Qiu, *Member, IEEE,* and Er-Wei Bai

***Abstract*—This paper proposes a general linear dual-rate structure for multirate signal processing, which encompasses the usual multirate building blocks—expanders, LTI filters, decimators, and their cascade combinations—as special cases. Structural properties of such dual-rate systems are studied in detail; in particular, several equivalent implementation structures are proposed: 1) cascade connections of a block expander, a linear periodically time-varying system, and a block decimator; 2) vector sample-rate changers; and 3) cascade connections of an expander, certain linear switching time-varying system, and a decimator. Using such general building blocks in multirate signal processing allows more design freedom and therefore can achieve what is otherwise impossible. This is illustrated in nonuniform multirate filter banks. Using the general building blocks as synthesis systems, the incompatibility for alias cancellation and structural dependency constraint for design, both due to fractional decimation ratios in different channels, are eliminated. Hence optimal design of synthesis systems is possible.**

***Index Terms*—Digital signal processing, fractional decimation, multirate filtering, nonuniform filter banks, optimization.**

I. INTRODUCTION

TRADITIONAL multirate building blocks in digital signal processing [9], [34] are decimators (downsamplers), expanders (upsamplers), and LTI filters, with possibly some summing junctions. An example is the fractional sample-rate changer shown in Fig. 1, where $\uparrow m$ is the expander by a factor m, $\downarrow n$ the decimator by n, and F a suitable LTI filter. The output sample rate is m/n times the input sample rate.

Such rate changers are not only useful in their own right [31], e.g., sample-rate conversion for bandlimited signals, they are also fundamental building blocks for multirate filter banks with uniform [9], [34] or nonuniform [27], [18] bands. Multirate signal processing has been studied a great deal (see [19], [1], [22], [34], [10], [36], [33] and their references).

The rate changer in Fig. 1 is in fact a dual-rate system with the input–output property that shifting the input (u) by n samples results in shifting the output (y) by m samples. Such a property is defined as (m, n)-shift invariance, which is a generalization of time invariance for single-rate systems—if $m = n = 1$, a linear, (m, n)-shift-invariant system reduces to an LTI system.

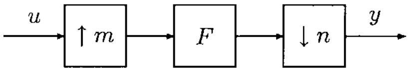

Fig. 1. Sample-rate changer.

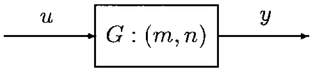

Fig. 2. A general dual-rate system.

Consider two classes of dual-rate systems: The class associated with the structure in Fig. 1, namely, the systems which are characterized by $\uparrow m$, LTI filter F, and $\downarrow n$, and the class of linear, causal, dual-rate systems shown in Fig. 2, in which the notation "G: (m, n)" means that G is (m, n)-shift-invariant. Is the latter more general than the former? Or, is it always possible to realize a causal, dual-rate linear system in Fig. 2 by the structure in Fig. 1 with the expander $\uparrow m$, some LTI F, and the decimator $\downarrow n$? The answer is positive if the two integers m and n are coprime and *negative* otherwise [30], in this case, m and n have some nontrivial common factor. As an simple example, if $m = n = 2$, a linear, (m, n)-shift-invariant system becomes a linear periodically time-varying (LPTV) system with period 2; but it can be shown that the structure in Fig. 1 in this case always represents an LTI system.

The class of dual-rate systems in Fig. 2 is more general than the structure in Fig. 1, and is especially interesting when m and n have common factors. In this paper, we study such (m, n)-shift-invariant systems in detail and use them as the fundamental building blocks for multirate systems. One main contribution of this paper is to show that such (m, n)-shift-invariant systems are realizable by using either block expanders and decimators and LPTV systems, or traditional expanders and decimators and certain linear switching time-varying systems.

Advantages of the more general setup have already been observed. Khansari and Leon-Garcia [16], [17] and Nayebi *et al.* [26] used LPTV filters and block decimators and expanders in filter bank systems; in particular, Khansari and Leon-Garcia [17] showed that using general synthesis systems in an FIR filter bank, perfect reconstruction is possible if and only if the analysis filters have no common zero (note that this condition is necessary but not sufficient for perfect reconstruction [37]

Manuscript received November 26, 1996; revised August 12, 1997. This work was supported by the Natural Sciences and Engineering Research Council of Canada, the Hong Kong Research Grants Council, and the National Science Foundation, under Grant ECS-9710297. A preliminary version of this paper was reported in *Proc. IEEE International Symposium on Circuits and Systems,* Hong Kong, June 1997.

T. Chen is with the Department of Electrical and Computer Engineering, University of Alberta, Edmonton, Alta., Canada T2N 1N4.

L. Qiu is with the Department of Electrical and Electronic Engineering, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong.

E. Bai is with the Department of Electrical and Computer Engineering, University of Iowa, Iowa City, IA 52242 USA.

Publisher Item Identifier S 1057-7130(98)04675-8.

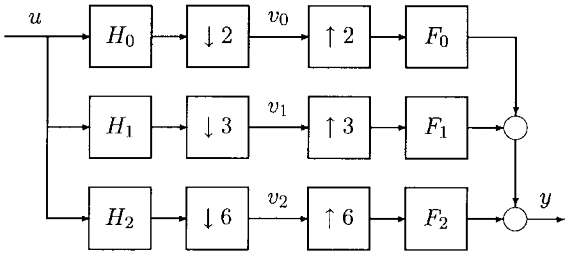

Fig. 3. A nonuniform filter bank.

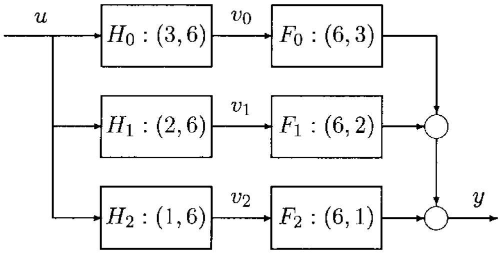

Fig. 4. A nonuniform filter bank with general structures.

using traditional building blocks); Xia and Suter [38], [39] later systematically studied block sampling in a more general setting using vector filter banks; Shenoy [30] showed that by using general structures one can achieve what is otherwise impossible in the design of fractional rate changers, and also pointed out that the structural dependency in designing nonuniform filter banks [18], [4], [5] disappears when general structures are used for analysis and synthesis.

As an application of the general dual-rate systems, consider the three-channel nonuniform filter bank shown in Fig. 3, where the analysis and synthesis filters H_i and F_i are all LTI and causal. Hoang and Vaidyanathan [14] showed that the decimation integers, $\{2, 3, 6\}$, form an incompatible set; i.e., it is impossible to achieve alias cancellation by designing the six filters, let alone perfect reconstruction. Also, a design difficulty called structural dependency arises in this setup [18], [4], [5]; this can be seen later after blocking the system to get the 6 × 6 equivalent transfer matrix.

However, if appropriate dual-rate systems are used for analysis and synthesis as shown in Fig. 4, the difficulties encountered using the structure in Fig. 3 no longer exist. This is because four of the six dual-rate filters used in Fig. 4—H_0, H_1, F_0, and F_1, have common factors in their m and n and hence are more general; they allow extra freedom in system design. In this case, optimal design based on model-matching theory, which was advocated in [31] for multirate filter design and in [6] for uniform filterbank design, can be accomplished with relative ease (detailed discussion to be given later).

Briefly, the paper is organized as follows. Section II defines precisely what we mean by dual-rate systems, and studies their fundamental properties such as shift invariance and causality; the blocking technique is used to convert the dual-rate systems into equivalent LTI systems. Section III shows that any linear dual-rate system can be realized by cascading a block expander, an LPTV system, and a block decimator, establishing the connection to block sampling used in [17]. Section IV gives a connection between dual-rate systems and vector decimators and expanders, and multivariable LTI systems, which were studied in vector filter banks in [38] and [39]. In Section V, we introduce a special class of LPTV systems called linear switching time-varying (LSTV) systems and prove the fact that any linear dual-rate system can be realized by cascading a traditional expander, an LSTV system, and a traditional decimator. Section VI applies the general structures to nonuniform filter banks, and shows that several limitations encountered when using traditional building blocks can be eliminated; an optimal design example is also given. Finally, in Section VII we offer some concluding remarks.

II. GENERAL DUAL-RATE SYSTEMS

Dual-rate systems are to be used as fundamental building blocks for multirate systems. In this section we study basic concepts of linear dual-rate systems such as shift invariance, causality, and their various representations.

Let ℓ be the space of discrete-time signals defined on the set of all integers. A signal x in ℓ is written

$$\{\cdots, x(-2), x(-1)|x(0), x(1), x(2), \cdots\}$$

the vertical bar separating the time from $k = 0$. A linear system G is regarded as a linear transformation mapping ℓ to itself, written $y = Gu$. It is a *dual-rate* system if the output and input have different sample rates—in this paper we shall make the assumption that the ratio of the two rates is a rational number, say, m/n; i.e., the output sample rate is m/n times the input sample rate.

A linear, dual-rate system G can always be represented by a kernel function $g(k, l)$

$$y = Gu \Leftrightarrow y(k) = \sum_l g(k, l)u(l), \qquad \forall k. \tag{1}$$

A. Shift Invariance

To define shift invariance precisely, let U be the unit time delay on ℓ with transfer function z^{-1}. For a linear, dual-rate system G with output and input sample-rate ratio m/n, we define G to be *(m, n)-shift-invariant* if

$$GU^n = U^m G.$$

(The two integers m and n need not be coprime.) This means that shifting the input by n samples results in shifting the output by m samples. In terms of the kernel functions, (m, n)-shift invariance is characterized by the following relation:

$$g(k+m, l+n) = g(k, l), \qquad \forall k, l.$$

Fig. 2 represents an (m, n)-shift-invariant G using block diagrams.

This shift invariance guarantees that appropriate blocking of the input and output gives rise to a multivariable (vector) LTI system [21], [24], [3], [25], [35]. (A similar blocking structure was also used earlier in convolutional codes [11].)

For an integer $p > 0$, define the p-fold *blocking operator*, L_p, via $\underline{x} = L_p x$ (underlining denotes blocking):

$$\begin{aligned}&\{\cdots \mid x(0),\, x(1),\, \cdots\}\\ &\quad\mapsto \left\{\cdots \left| \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(p-1) \end{bmatrix}, \begin{bmatrix} x(p) \\ x(p+1) \\ \vdots \\ x(2p-1) \end{bmatrix}, \cdots \right.\right\}. \qquad (2)\end{aligned}$$

L_p maps ℓ to ℓ^p, the external direct sum of p copies of ℓ. If the underlying period for x is h, the underlying period for the blocked signal $\underline{x}$ is ph. The inverse L_p^{-1}, mapping ℓ^p to ℓ, amounts to reversing the operation in (2).

Let G be linear, dual-rate, and SISO (single-input, single-output). Block the input and output appropriately to get the blocked system $\underline{G} := L_m G L_n^{-1}$, which has n inputs and m outputs. It is a well-known fact that $\underline{G}$ is LTI iff G is $(m,\, n)$-shift-invariant [21], [25], [35]. Hence, if G is $(m,\, n)$-shift-invariant, $\underline{G}$ has an $m \times n$ transfer matrix:

$$\underline{\hat{G}}(z) = \begin{bmatrix} \hat{G}_{00}(z) & \hat{G}_{01}(z) & \cdots & \hat{G}_{0,\,n-1}(z) \\ \hat{G}_{10}(z) & \hat{G}_{11}(z) & \cdots & \hat{G}_{1,\,n-1}(z) \\ \vdots & \vdots & & \vdots \\ \hat{G}_{m-1,\,0}(z) & \hat{G}_{m-1,\,1}(z) & \cdots & \hat{G}_{m-1,\,n-1}(z) \end{bmatrix}.$$

The entries in this matrix relate to the kernel function $g(k,\, l)$ of G as follows:

$$\hat{G}_{ij}(z) = \sum_k g(i + km,\, j) z^{-k}. \qquad (3)$$

B. Causality

Causality of a dual-rate system G reflects implementability of the system in real time. Let G have input u and output y. Because the ratio of the sample rates of y and u is m/n, we can take the sample periods of u and y to be mh and nh, respectively, where h is some real number. Assuming both u and y are synchronized at time $t = 0$, we have that $u(k)$ occurs at time $t = k(mh)$ and $y(k)$ at $t = k(nh)$. Thus, G is *causal* if for any k, the output $y(k)$ depends only on inputs occurred at $t \le k(nh)$, or on $u(l)$ for all l satisfying $lm \le kn$. Similarly, G is *strictly causal* if $y(k)$ depends only on $u(l)$ for all l such that $lm < kn$.

In terms of the kernel function in (1), G is causal iff

$$g(k,\, l) = 0 \quad \text{whenever} \quad kn < lm \qquad (4)$$

and is strictly causal iff

$$g(k,\, l) = 0 \quad \text{whenever} \quad kn \le lm.$$

If the linear, dual-rate system G is both $(m,\, n)$-shift-invariant and causal, the blocked system $\underline{G}$ is LTI and causal. Moreover, the direct feedthrough matrix

$$\underline{\hat{G}}(\infty) = \begin{bmatrix} D_{00} & D_{01} & \cdots & D_{0,\,n-1} \\ D_{10} & D_{11} & \cdots & D_{1,\,n-1} \\ \vdots & \vdots & & \vdots \\ D_{m-1,\,0} & D_{m-1,\,1} & \cdots & D_{m-1,\,n-1} \end{bmatrix} \qquad (5)$$

must have a block lower-triangular structure due to causality. This is called causality constraint [20], [7] and can be derived from (4) and (3):

$$D_{ij} = 0 \quad \text{whenever} \quad in < jm. \qquad (6)$$

As an example, consider the rate changer in Fig. 1 with $m = 2$, $n = 3$, and F being LTI and causal. It follows that the overall (dual-rate) system is (2, 3)-shift-invariant and causal. Introduce the 6-fold, type-1 polyphase decomposition [34] of $\hat{F}(z)$,

$$\hat{F}(z) = \sum_{i=0}^{5} z^{-i} \hat{F}_i(z^6)$$

the polyphase components $\hat{F}_i(z)$ being causal. Then the blocked dual-rate system has the transfer matrix [15], [4], [5]

$$\begin{bmatrix} \hat{F}_0(z) & z^{-1}\hat{F}_4(z) & z^{-1}\hat{F}_2(z) \\ \hat{F}_3(z) & \hat{F}_1(z) & z^{-1}\hat{F}_5(z) \end{bmatrix}. \qquad (7)$$

Note that this matrix contains all six polyphase components due to the coprimeness of m and n (to be seen later); note also that the (0, 1), (0, 2), and (1, 2) entries are all strictly causal, which shows causality of the dual-rate system.

From now on, we shall restrict our attention to linear, dual-rate systems which are shift-invariant, causal, and, moreover, whose blocked transfer matrices have real-rational functions of z as their entries. All such transfer matrices have state-space models

$$x(k+1) = Ax(k) + B\underline{u}(k) \qquad (8)$$

$$\underline{y}(k) = Cx(k) + D\underline{u}(k) \qquad (9)$$

where x is the state vector, and A, B, C, D are all real matrices of compatible dimensions with A square. The matrices B, C, D are partitioned according to the input and output dimensions of $\underline{G}$:

$$B = [B_0 \quad B_1 \quad \cdots \quad B_{n-1}], \qquad C = \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_{m-1} \end{bmatrix}$$

and D is the same as the right-hand matrix in (5) satisfying the causality constraint in (6).

The state-space model in (8), (9) is for the blocked system. The corresponding difference equations for the original dual-rate system are as follows:

$$\begin{aligned} x(k+1) &= Ax(k) + \sum_{j=0}^{n-1} B_j u(kn + j) \\ y(km + i) &= C_i x(k) + \sum_{j=0}^{n-1} D_{ij} u(kn + j) \\ &\qquad i = 0,\, 1,\, \cdots,\, m-1. \end{aligned}$$

These equations can be used to implement the dual-rate system: The input u is updated every mh seconds, the vector x every mnh seconds, and the output y every nh seconds.

Such systems are implementable on, e.g., general-purpose DSP cards, with only finite memory.

The state-space model in (8), (9) is general and holds for both IIR and FIR systems. If $\underline{G}$ is FIR with order, say, p, a simpler model is given by

$$\underline{y}(k) = M_0\underline{u}(k) + M_1\underline{u}(k-1) + \cdots + M_p\underline{u}(k-p)$$

where $M_0, M_1, \cdots, M_p$ are all $m \times n$ constant matrices, and M_0 satisfies the condition in (6) for causality. From here on, one can also write down the corresponding equations for the original system relating y to u.

In this paper, we use the blocking technique consistently for treating dual-rate systems. An alternative technique is based on alias-component (AC) matrices in the frequency domain; we refer to Shenoy [29], [30], Shenoy *et al.* [31], and Reng [28] for recent related studies.

Finally, we make some remarks on using dual-rate systems as building blocks for multirate signal processing systems. The class of dual-rate systems we are interested are SISO, causal, and (m, n)-shift-invariant, where m and n range over all positive integers. Traditional building blocks are decimators, expanders, and LTI filters, which are all SISO. They are all special cases of dual-rate systems: The decimator $\downarrow n$ is $(1, n)$-shift-invariant and causal; the expander $\uparrow m$ is $(m, 1)$-shift-invariant and causal; any LTI filter is (1, 1)-shift-invariant. However, general LPTV systems with period m are dual-rate systems [they are (m, m)-shift-invariant] but cannot be constructed using decimators, expanders, and *one* SISO, LTI filter. In this sense, we conclude that dual-rate systems are more general building blocks for multirate systems.

III. REALIZATION VIA BLOCK DECIMATION AND EXPANSION

In this section, we establish the connection between dual-rate systems discussed in the preceding section and the generalized elements: block decimators, block expanders, and LPTV systems; these generalized elements have already been studied in multirate filter banks [16], [17], [26], [38]. The result in this section also provides a way for realizing general dual-rate systems.

Let us first consider the special case when m and n are coprime. Then the structures in Figs. 1 and 2 are equivalent [30]:

Lemma 1: If m and n are coprime, any linear, (m, n)-shift-invariant system in Fig. 2 is realizable by the cascade structure in Fig. 1 of the expander $\uparrow m$, an LTI filter F, and the decimator $\downarrow n$.

This is already a known result, which was proven, e.g., as a lemma in [5] from a different perspective.

A linear, dual-rate system which is (m, n)-shift-invariant cannot be represented by the structure in Fig. 1 for some LTI F, if m and n have nontrivial common factor [30]. In this case, as we will show, block decimators and expanders and LPTV systems should be used instead.

Two positive integers p and q characterize the *block decimator* shown in Fig. 5, where q represents the block size and p the decimation ratio. For any integer k, the input–output equation is

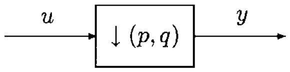

Fig. 5. The block decimator.

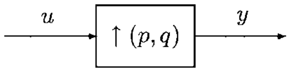

Fig. 6. The block expander.

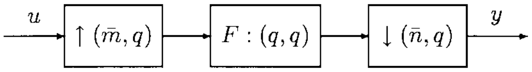

Fig. 7. An equivalent structure for dual-rate systems.

$$y(kq+i) = u(kpq+i), \qquad i = 0, 1, \cdots, q-1.$$

If one groups the input into blocks of size q (starting from time $k = 0$), this block decimator retains only every pth block. It can be verified that the block decimator is (q, pq)-shift-invariant and causal; moreover, if $q = 1$, it reduces to the traditional decimator: $\downarrow (p, 1) = \downarrow p$.

The *block expander* is the dual system, shown in Fig. 6, where q again is the block size and p the expansion ratio. For any integer k, the block expander is defined via

$$y(kpq+i) = \begin{cases} u(kq+i), & i = 0, 1, \cdots, q-1, \\ 0, & i = q, q+1, \cdots, pq-1. \end{cases}$$

This corresponds to inserting $p-1$ blocks of zeros if the input and output are blocked with size q. The block expander is (pq, q)-shift-invariant but noncausal, in general. If $q = 1$, it reduces to the traditional expander $\uparrow p$. (For the frequency-domain input–output relations, see [17], [38].)

Let G be any linear, dual-rate system which is (m, n)-shift-invariant. If m and n are not coprime, we can find the largest common factor q and write $m = \overline{m}q$, $n = \overline{n}q$, so that $\overline{m}$ and $\overline{n}$ are coprime. Thus we can state the main result of this section.

Theorem 1: The dual-rate system G is realizable by the cascade structure, shown in Fig. 7, of the block expander $\uparrow (\overline{m}, q)$, a single-rate, LPTV system F with period q, and the block decimator $\downarrow (\overline{n}, q)$.

Note that if m and n are coprime, then $q = 1$. In this case, the structure in Fig. 7 reduces to that in Fig. 1 with F LTI; and the theorem reduces to Lemma 1.

Proof of Theorem 1: In the following we shall outline the main steps in the proof. First it can be verified that the overall system in Fig. 7, $S : u \mapsto y$, is dual-rate, (m, n)-shift-invariant. To show that this structure can be used to represent any dual-rate, (m, n)-shift-invariant system G, it suffices to show that by proper choice of the LPTV F, the transfer matrix for the blocked system $\underline{S} := L_m S L_n^{-1}$ can match any given $m \times n$ transfer matrix $\hat{\underline{G}}(z)$ (the blocked system for G); or equivalently, the mn entries in $\hat{\underline{S}}(z)$ can be arbitrarily selected.

For ease of reference in equations, let us write the block expander in Fig. 7 as $E_{\overline{m},q}$ and the block decimator $D_{\overline{n},q}$. Then $S = D_{\overline{n},q} F E_{\overline{m},q}$. In order to find $\underline{S}$, note first that

$\underline{F} := L_q F L_q^{-1}$ is LTI with q inputs and outputs:

$$\underline{F} = \begin{bmatrix} F_{00} & \cdots & F_{0,\,q-1} \\ \vdots & & \vdots \\ F_{q-1,\,0} & \cdots & F_{q-1,\,q-1} \end{bmatrix}. \tag{10}$$

All subsystems F_{ij} are LTI and freely assignable. The following identities follow readily:

$$L_q D_{\overline{n},\,q} L_q^{-1} = \begin{bmatrix} D_{\overline{n}} & & \\ & \ddots & \\ & & D_{\overline{n}} \end{bmatrix}_{q\times q}$$
$$L_q E_{\overline{m},\,q} L_q^{-1} = \begin{bmatrix} E_{\overline{m}} & & \\ & \ddots & \\ & & E_{\overline{m}} \end{bmatrix}_{q\times q} \tag{11}$$

where $D_{\overline{n}}$ is simply $\downarrow \overline{n}$ and $E_{\overline{m}}$ is $\uparrow \overline{m}$. Let $P_{\overline{m},\,q}$ be the $m \times m$ permutation matrix which takes the (column) vector

$$[0 \quad q \quad \cdots \quad (\overline{m}-1)q \quad 1 \quad q+1 \quad \cdots \quad (\overline{m}-1)q+1 \quad \cdots \quad q-1 \quad 2q-1 \quad \cdots \quad \overline{m}q-1]'$$

(prime denotes matrix transpose) to the vector

$$[0 \quad 1 \quad 2 \quad \cdots \quad \overline{m}q-1]'$$

similarly for $P_{\overline{n},\,q}$. Then it can be shown that

$$L_m = P_{\overline{m},\,q} \begin{bmatrix} L_{\overline{m}} & & \\ & \ddots & \\ & & L_{\overline{m}} \end{bmatrix} L_q,$$
$$L_n^{-1} = L_q^{-1} \begin{bmatrix} L_{\overline{n}}^{-1} & & \\ & \ddots & \\ & & L_{\overline{n}}^{-1} \end{bmatrix} P_{\overline{n},\,q}^{-1}. \tag{12}$$

Based on (10)–(12), we can write

$$\begin{aligned} \underline{S} &= L_m D_{\overline{n},\,q} F E_{\overline{m},\,q} L_n^{-1} \\ &= L_m L_q^{-1} (L_q D_{\overline{n},\,q} L_q^{-1})(L_q F L_q^{-1})(L_q E_{\overline{m},\,q} L_q^{-1}) L_q L_n^{-1} \\ &= P_{\overline{m},\,q} \begin{bmatrix} L_{\overline{m}} D_{\overline{n}} & & \\ & \ddots & \\ & & L_{\overline{m}} D_{\overline{n}} \end{bmatrix} \begin{bmatrix} F_{00} & \cdots & F_{0,\,q-1} \\ \vdots & & \vdots \\ F_{q-1,\,0} & \cdots & F_{q-1,\,q-1} \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} E_{\overline{m}} L_{\overline{n}}^{-1} & & \\ & \ddots & \\ & & E_{\overline{m}} L_{\overline{n}}^{-1} \end{bmatrix} P_{\overline{n},\,q}^{-1}. \end{aligned}$$

From here, the inner three matrices together can be viewed as a $q \times q$ block system, the $(i,\,j)$ block being

$$L_{\overline{m}} D_{\overline{n}} F_{ij} E_{\overline{m}} L_{\overline{n}}^{-1}$$

which is $\overline{m} \times \overline{n}$ and whose entries are arbitrarily assignable by proper choice of the LTI F_{ij} because $\overline{m}$ and $\overline{n}$ are coprime (Lemma 1). Hence, the inner three matrices together can match any given transfer matrix; so can $\underline{S}$ because $P_{\overline{m},\,q}$ and $P_{\overline{n},\,q}$ are permutation matrices. Thus, the proof is completed. ☐

The proof supports a procedure to compute the LPTV F if the dual-rate system G is given. To illustrate, consider the case with $m = 4$ and $n = 6$. Given the 4×6 blocked transfer matrix $[\hat{G}_{ij}(z)]$, we want to realize G via the structure in Fig. 7 with $\overline{m} = 2$, $\overline{n} = 3$, and $q = 2$, or, more precisely, find the LPTV F with period 2. Write the blocked F ($\underline{F} = L_2 F L_2^{-1}$) as

$$\hat{\underline{F}}(z) = \begin{bmatrix} \hat{F}_{00}(z) & \hat{F}_{01}(z) \\ \hat{F}_{10}(z) & \hat{F}_{11}(z) \end{bmatrix}.$$

Then from the proof of Theorem 1 we set

$$\underline{G} = P_{2,\,2} \underline{R} P_{3,\,2}^{-1} \tag{13}$$

with

$$\begin{aligned} \underline{R} &= \begin{bmatrix} R_{00} & R_{01} \\ R_{10} & R_{11} \end{bmatrix} \\ &:= \begin{bmatrix} L_2 D_3 & 0 \\ 0 & L_2 D_3 \end{bmatrix} \begin{bmatrix} F_{00} & F_{01} \\ F_{10} & F_{11} \end{bmatrix} \begin{bmatrix} E_2 L_3^{-1} & 0 \\ 0 & E_2 L_3^{-1} \end{bmatrix}. \end{aligned}$$

Since $R_{ij} = L_2 D_3 F_{ij} E_2 L_3^{-1}$, its transfer matrix is of the form in (7) in terms of type-1 polyphase components [34] of $\hat{F}_{ij}(z)$, denoted $\hat{F}_{ij}^{k}(z)$ for $k = 0,\, 1,\, \cdots,\, 5$. Equation (13) is equivalent to $P_{2,\,2}^{-1} \underline{G} P_{3,\,2} = \underline{R}$, as shown in (14) at the bottom the page. Hence,

$$\begin{aligned} \hat{F}_{00}(z) &= \hat{F}_{00}^{0}(z^6) + z^{-1}\hat{F}_{00}^{1}(z^6) + z^{-2}\hat{F}_{00}^{2}(z^6) + z^{-3}\hat{F}_{00}^{3}(z^6) \\ &\quad + z^{-4}\hat{F}_{00}^{4}(z^6) + z^{-5}\hat{F}_{00}^{5}(z^6) \\ &= \hat{G}_{00}(z^6) + z^{-1}\hat{G}_{22}(z^6) + z^{4}\hat{G}_{04}(z^6) + z^{-3}\hat{G}_{20}(z^6) \\ &\quad + z^{2}\hat{G}_{02}(z^6) + z\hat{G}_{24}(z^6). \end{aligned} \tag{15}$$

This relates F_{00} to the given dual-rate system G in terms of its blocked transfer matrix. We remark that if G is causal, so is F_{00}. Causality of G requires that $\hat{G}_{04}(z)$, $\hat{G}_{02}(z)$, and $\hat{G}_{24}(z)$ be *strictly* causal; hence the terms such as $z^4\hat{G}_{04}(z^6)$ in (14) are still causal. Similarly, one can determine other $\hat{F}_{ij}(z)$ from their polyphase components and (14).

$$\begin{aligned} &\begin{bmatrix} \hat{G}_{00}(z) & \hat{G}_{02}(z) & \hat{G}_{04}(z) & \hat{G}_{01}(z) & \hat{G}_{03}(z) & \hat{G}_{05}(z) \\ \hat{G}_{20}(z) & \hat{G}_{22}(z) & \hat{G}_{24}(z) & \hat{G}_{21}(z) & \hat{G}_{23}(z) & \hat{G}_{25}(z) \\ \hat{G}_{10}(z) & \hat{G}_{12}(z) & \hat{G}_{14}(z) & \hat{G}_{11}(z) & \hat{G}_{13}(z) & \hat{G}_{15}(z) \\ \hat{G}_{30}(z) & \hat{G}_{32}(z) & \hat{G}_{34}(z) & \hat{G}_{31}(z) & \hat{G}_{33}(z) & \hat{G}_{35}(z) \end{bmatrix} \\ &\quad = \begin{bmatrix} \hat{F}_{00}^{0}(z) & z^{-1}\hat{F}_{00}^{4}(z) & z^{-1}\hat{F}_{00}^{2}(z) & \hat{F}_{01}^{0}(z) & z^{-1}\hat{F}_{01}^{4}(z) & z^{-1}\hat{F}_{01}^{2}(z) \\ \hat{F}_{00}^{3}(z) & \hat{F}_{00}^{1}(z) & z^{-1}\hat{F}_{00}^{5}(z) & \hat{F}_{01}^{3}(z) & \hat{F}_{01}^{1}(z) & z^{-1}\hat{F}_{01}^{5}(z) \\ \hat{F}_{10}^{0}(z) & z^{-1}\hat{F}_{10}^{4}(z) & z^{-1}\hat{F}_{10}^{2}(z) & \hat{F}_{11}^{0}(z) & z^{-1}\hat{F}_{11}^{4}(z) & z^{-1}\hat{F}_{11}^{2}(z) \\ \hat{F}_{10}^{3}(z) & \hat{F}_{10}^{1}(z) & z^{-1}\hat{F}_{10}^{5}(z) & \hat{F}_{11}^{3}(z) & \hat{F}_{11}^{1}(z) & z^{-1}\hat{F}_{11}^{5}(z) \end{bmatrix} \end{aligned} \tag{14}$$

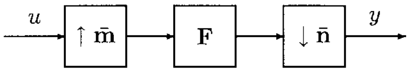

Fig. 8. The vector sample-rate changer.

IV. REALIZATION VIA VECTOR SAMPLE-RATE CHANGERS

Theorem 1 also establishes a connection through blocking between dual-rate systems and vector sample-rate changers studied by Xia and Suter [38], [39]. For any dual-rate system G which is (m, n)-shift-invariant, introduce $\overline{m}$, $\overline{n}$, and q as in Theorem 1. Define a vector sample-rate changer as in Fig. 8, where $\uparrow \overline{\boldsymbol{m}}$ is the vector expander and $\downarrow \overline{\boldsymbol{n}}$ the vector decimator, both with q channels

$$\uparrow \overline{\boldsymbol{m}} = \begin{bmatrix} \uparrow \overline{m} & & \\ & \ddots & \\ & & \uparrow \overline{m} \end{bmatrix}, \qquad \downarrow \overline{\boldsymbol{n}} = \begin{bmatrix} \downarrow \overline{n} & & \\ & \ddots & \\ & & \downarrow \overline{n} \end{bmatrix}$$

and $\boldsymbol{F}$ is a multivariable (vector) LTI system with q inputs and q outputs. The structure in Fig. 8 relates to the dual-rate system G through q-fold blocking.

Theorem 2: The blocked dual-rate system, $L_q G L_q^{-1}$, is realizable by the structure in Fig. 8 for some LTI $\boldsymbol{F}$.

Proof: By Theorem 1, G can be represented by the structure in Fig. 7 for some LPTV F with period q: $G = D_{\overline{n},q} F E_{\overline{m},q}$, following the notation in the proof of Theorem 1. Thus,

$$\begin{aligned} L_q G L_q^{-1} &= L_q D_{\overline{n},q} F E_{\overline{m},q} L_q^{-1} \\ &= (L_q D_{\overline{n},q} L_q^{-1})(L_q F L_q^{-1})(L_q E_{\overline{m},q} L_q^{-1}) \\ &= \begin{bmatrix} \downarrow \overline{n} & & \\ & \ddots & \\ & & \downarrow \overline{n} \end{bmatrix} (L_q F L_q^{-1}) \begin{bmatrix} \uparrow \overline{m} & & \\ & \ddots & \\ & & \uparrow \overline{m} \end{bmatrix} \end{aligned}$$

The last equality follows from (11). Identifying $\boldsymbol{F} = L_q F L_q^{-1}$ which is LTI because F is LPTV with period q, completes the proof. □

The converse of Theorem 2 is also true: one can represent the vector sample-rate changer in Fig. 8 through blocking a dual-rate system. Thus, the two structures are equivalent in this sense.

Connecting with Theorem 1, we also conclude that the structure in Fig. 7 based on block sampling and the one in Fig. 8 based on vector processing are equivalent through blocking. At this point, one may wonder why Xia and Suter [38] showed that their vector filter banks, which are built with structures in Fig. 8, are more general than those in [17] using block sampling. This is because LTI filters were used in [17] for constructing the filter banks. Had LPTV filters been used appropriately according to Theorem 1, the two filter bank structures would have been equivalent.

V. REALIZATION VIA LINEAR SWITCHING TIME-VARYING SYSTEMS

In the preceding two sections, we presented two structures which can be used to implement a general dual-rate system: The first one uses block decimation and expansion, and the second uses vector decimation and expansion. The problem with the first is that real-time implementation of the block decimator and expander is technically demanding: The block decimator requires certain storage function and the block expander is noncausal, if the input and output samples are uniform in time. (These problems seem to disappear if input and output samples are allowed to be nonuniform in time.) The disadvantage of the second structure is that blocking is required to convert an SISO system to a multivariable (vector) system.

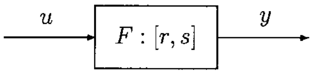

Fig. 9. An $[r, s]$-LSTV system.

If m and n are already coprime, Lemma 1 states that any (m, n)-shift-invariant system can be implemented by the structure in Fig. 1 using only the traditional decimator and expander. If m and n are not coprime, can we make use of a similar structure with traditional decimator and expander and some SISO system F? The answer is positive; but we need to consider a special class of LPTV systems for F, which are called linear switching time-varying (LSTV) systems. With these LSTV structures, the above mentioned disadvantages associated with Figs. 7 and 8 are eliminated.

For two positive integers r and s, an $[r, s]$-LSTV system F, represented in Fig. 9, consists of r LTI subsystems, $F_0, F_1, \cdots, F_{r-1}$, and a switching device as depicted in Fig. 10. The switching device switches the input u to each subsystem for exactly s samples starting from F_0 to F_{r-1} and then repeats. In other words, divide the time into intervals of rs samples starting from $k = 0$; during the first interval, $0 \le k \le rs - 1$, the inputs $u(k)$ over $js \le k \le (j+1)s - 1$ $(0 \le j \le r-1)$ are connected to F_j. Similarly for the remaining intervals, the outputs of all subsystems are summed up to form the overall output y. Note that at times when one subsystem is switched to u, the other subsystems are assumed to have zero inputs, and their outputs evolve according to the past history. This LSTV structure can be implemented with the periodic switching device and the LTI subsystems; for example, it is possible to implement them on microprocessors.

Let $f(k, l)$ be the kernel representation of the LSTV F:

$$y = Fu \Leftrightarrow y(k) = \sum_l f(k, l) u(l), \qquad \forall k.$$

Let $f_j(k)$ be the impulse response of the LTI subsystem F_j. Now think of $[f(k, l)]$ as an infinite matrix; the infinite matrix representation of F_j is $[f_j(k-l)]$, a Toeplitz matrix. The switching mechanism implies that $[f(k, l)]$ can be obtained from $[f_j(k-l)]$ as follows. Starting from $l = 0$, the first s columns of $[f(k, l)]$ are identical to the first s columns in $[f_0(k-l)]$, the next s columns are identical to the corresponding ones in $[f_1(k-l)]$, and so on; and repeat this process for every rs columns in $[f(k, l)]$. In equations, we write

$$\begin{aligned} f(k, l) &= f_j(k-l), \\ l &= js, js+1, \cdots, js+s-1 \bmod rs. \end{aligned} \tag{16}$$

It follows readily from (16) that the LSTV system F is LPTV with period rs. Hence, $\underline{F} := L_{rs} F L_{rs}^{-1}$ is LTI with

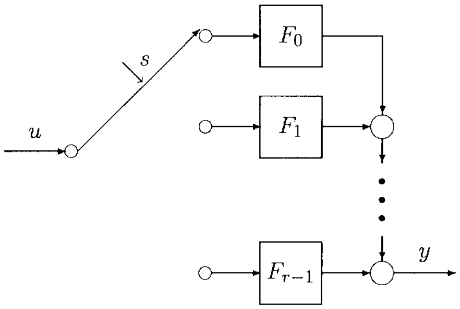

Fig. 10. The structure of an $[r, s]$-LSTV system.

an $(rs) \times (rs)$ transfer matrix. To see how to write down this transfer matrix, let us consider the case with $r = s = 2$; then $\hat{\underline{F}}(z)$ is 4×4. Introducing the type-1 polyphase representations [34] for F_0 and F_1,

$$\hat{F}_j(z) = \sum_{i=0}^{3} z^{-i} \hat{F}_j^i(z^4), \qquad j = 0, 1,$$

we get the blocked systems $\underline{F}_j = L_4 F_j L_4^{-1}$ ($j = 0, 1$) as pseudocirculant matrices

$$\hat{\underline{F}}_j(z) = \begin{bmatrix} \hat{F}_j^0(z) & z^{-1}\hat{F}_j^3(z) & z^{-1}\hat{F}_j^2(z) & z^{-1}\hat{F}_j^1(z) \\ \hat{F}_j^1(z) & \hat{F}_j^0(z) & z^{-1}\hat{F}_j^3(z) & z^{-1}\hat{F}_j^2(z) \\ \hat{F}_j^2(z) & \hat{F}_j^1(z) & \hat{F}_j^0(z) & z^{-1}\hat{F}_j^3(z) \\ \hat{F}_j^3(z) & \hat{F}_j^2(z) & \hat{F}_j^1(z) & \hat{F}_j^0(z) \end{bmatrix}, \quad j = 0, 1.$$

Then it follows from (3) and (16) that $\hat{\underline{F}}(z)$ can be formed by putting together the first two columns of $\hat{\underline{F}}_0(z)$ and the last two columns of $\hat{\underline{F}}_1(z)$:

$$\hat{\underline{F}}(z) = \begin{bmatrix} \hat{F}_0^0(z) & z^{-1}\hat{F}_0^3(z) & z^{-1}\hat{F}_1^2(z) & z^{-1}\hat{F}_1^1(z) \\ \hat{F}_0^1(z) & \hat{F}_0^0(z) & z^{-1}\hat{F}_1^3(z) & z^{-1}\hat{F}_2^2(z) \\ \hat{F}_0^2(z) & \hat{F}_0^1(z) & \hat{F}_1^0(z) & z^{-1}\hat{F}_1^3(z) \\ \hat{F}_0^3(z) & \hat{F}_0^2(z) & \hat{F}_1^1(z) & \hat{F}_1^0(z) \end{bmatrix}.$$

Note that if this matrix is partitioned into two 4×2 submatrices, each submatrix is pseudocirculant column-wise but not row-wise.

This observation generalizes in the obvious way for the LSTV structure in Fig. 10.

Lemma 2: Let $\underline{F}_j = L_{rs} F_j L_{rs}^{-1}$; partition each transfer matrix as follows:

$$\hat{\underline{F}}_j(z) = \begin{bmatrix} \hat{\underline{F}}_j^0(z) & \hat{\underline{F}}_j^1(z) & \cdots & \hat{\underline{F}}_j^{r-1}(z) \end{bmatrix}$$

every submatrix being $(rs) \times s$. Then the transfer matrix for the blocked LSTV system, $\underline{F} = L_{rs} F L_{rs}^{-1}$, is given by

$$\hat{\underline{F}}(z) = \begin{bmatrix} \hat{\underline{F}}_0^0(z) & \hat{\underline{F}}_1^1(z) & \cdots & \hat{\underline{F}}_{r-1}^{r-1}(z) \end{bmatrix}.$$

Each submatrix in $\underline{F}(z)$ is pseudocirculant column-wise only.

Now we are set up to state and prove the main result of this section: any general dual-rate system can be implemented by an LSTV system along with some traditional decimator and expander.

Let G be any linear, dual-rate system which is (m, n)-shift-invariant; write $m = \overline{m}q$ and $n = \overline{n}q$ for some integer factor q so that $\overline{m}$ and $\overline{n}$ are coprime.

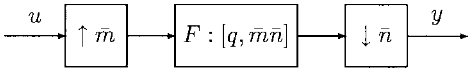

Fig. 11. An equivalent structure for dual-rate systems.

Theorem 3: The dual-rate system is realizable by the cascade structure, shown in Fig. 11, of the expander $\uparrow \overline{m}$, some $[q, \overline{m}\,\overline{n}]$-LSTV system F, and the decimator $\downarrow \overline{n}$.

Proof: It is easy to check that $H := D_{\overline{n}} F E_{\overline{m}}$ is dual-rate, (m, n)-shift-invariant, where $D_{\overline{n}} = \downarrow \overline{n}$ and $E_{\overline{m}} = \uparrow \overline{m}$. To show that this structure can be used to realize any (m, n)-shift-invariant system G, it suffices to show that the blocked system, $\underline{H} = L_m H L_n^{-1}$, has an $m \times n$ transfer matrix whose elements are all freely assignable by proper choice of F. Now

$$\underline{H} = L_m D_{\overline{n}} F E_{\overline{m}} L_n^{-1}. \tag{17}$$

To relate this to the blocked system of F in Lemma 2, define two linear transformations $K_{\overline{n}}$ and $J_{\overline{m}}$ as block diagonal matrices:

$$\begin{aligned} K_{\overline{n}} &= \text{block diag}\{[1 \quad 0 \quad \cdots \quad 0]_{1\times\overline{n}}, \cdots, \\ &\qquad [1 \quad 0 \quad \cdots \quad 0]_{1\times\overline{n}}\} \\ J_{\overline{m}} &= \text{block diag}\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{\overline{m}\times 1}, \cdots, \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{\overline{m}\times 1} \right\}. \end{aligned}$$

The blocks in $K_{\overline{n}}$ are repeated m times, and the blocks in $J_{\overline{m}}$ n times. Then it can be verified that

$$L_m D_{\overline{n}} = K_{\overline{n}} L_{m\overline{n}}, \; E_{\overline{m}} L_n^{-1} = L_{\overline{m}n}^{-1} J_{\overline{m}}. \tag{18}$$

Equations (17) and (18) lead to

$$\begin{aligned} \underline{H} &= K_{\overline{n}} (L_{\overline{m}\,\overline{n}q} F L_{\overline{m}\,\overline{n}q}^{-1}) J_{\overline{m}} \\ &= K_{\overline{n}} \underline{F} J_{\overline{m}} \end{aligned}$$

where $\underline{F}$ has an $(\overline{m}\,\overline{n}q) \times (\overline{m}\,\overline{n}q)$ transfer matrix $\hat{\underline{F}}(z)$ studied in Lemma 2.

Note that pre- and post-multiplying $\hat{\underline{F}}(z)$ by $K_{\overline{n}}$ and $J_{\overline{m}}$, respectively, amount to extracting an $m \times n$ matrix from $\hat{\underline{F}}(z)$ by taking rows numbered $0, \overline{n}, 2\overline{n}, \cdots, (m-1)\overline{n}$, and columns numbered $0, \overline{m}, 2\overline{m}, \cdots, (n-1)\overline{m}$. Since $\hat{\underline{F}}(z)$ can be partitioned into column pseudocirculant submatrices (Lemma 2)

$$\hat{\underline{F}}(z) = \begin{bmatrix} \hat{\underline{F}}_0^0(z) & \hat{\underline{F}}_1^1(z) & \cdots & \hat{\underline{F}}_{q-1}^{q-1}(z) \end{bmatrix}$$

$\hat{\underline{H}}(z)$ has a corresponding partition

$$\hat{\underline{H}}(z) = \begin{bmatrix} \hat{\underline{H}}_0(z) & \hat{\underline{H}}_1(z) & \cdots & \hat{\underline{H}}_{q-1}(z) \end{bmatrix} \tag{19}$$

the blocks being mutually independent. Note that $\hat{\underline{H}}_j(z)$ relates to $\hat{\underline{F}}_j^j(z)$, which in turn relates to F_j. Now it remains to show that all elements in $\hat{\underline{H}}_j(z)$ are assignable by proper choice of F_j.

Take $\hat{\underline{H}}_0(z)$ as an example; this is an $m \times \overline{n}$ matrix whose elements are those in $\hat{\underline{F}}_0^0(z)$ with the following row and column index pairs:

$$(i\overline{n}, j\overline{m}) : i = 0, 1, \cdots, m-1, j = 0, 1, \cdots, \overline{n} - 1.$$

Because all elements in the same column of $\underline{\hat{F}}_0^0(z)$ are independent (Lemma 2), if two elements of $\underline{\hat{H}}_0(z)$, say, the $(i_1\overline{n},\, j_1\overline{m})$th and $(i_2\overline{n},\, j_2\overline{m})$th elements in $\underline{\hat{F}}_0^0(z)$, were related, it would be due to the column pseudo-circulant property; in this case,

$$i_1\overline{n} - j_1\overline{m} = i_2\overline{n} - j_2\overline{m} \bmod (\overline{m}\,\overline{n}q).$$

Thus, we can write

$$(i_1 - i_2)\overline{n} + (j_2 - j_1)\overline{m} = k(\overline{m}\,\overline{n}q) \tag{20}$$

$$0 \le i_1,\, i_2 \le m - 1 \tag{21}$$

$$0 \le j_1,\, j_2 \le \overline{n} - 1 \tag{22}$$

k in (20) being an integer. The fact that $\overline{m}$ and $\overline{n}$ are coprime and (20) imply that $\overline{n}$ divides $j_2 - j_1$; then from (22) we have $j_1 = j_2$. In this case, (20) reduces to

$$(i_1 - i_2)\overline{n} = k\overline{m}\,\overline{n}q$$

which implies that $m = \overline{m}q$ divides $i_1 - i_2$, which implies with (21) in turn that $i_1 = i_2$.

The above shows that the elements in $\underline{\hat{H}}_0(z)$ are all independent; similarly for the other blocks in (19). □

To get the $(\overline{m}\,\overline{n}q)$-fold blocked system $\underline{F}$, according to Lemma 2, we need the $(\overline{m}\,\overline{n}q)$-fold polyphase components of the subsystems $F_j,\ j = 0, 1, \cdots, q-1$. Every one of those polyphase components appear exactly once in $\underline{\hat{H}}(z)$. Hence, if the structure in Fig. 11 is used to realize a dual-rate, (m, n)-shift-invariant system G, there exists a one-to-one correspondence between the dual-rate G and the set of subsystems of F, $\{F_0, F_1, \cdots, F_{q-1}\}$. A procedure outlining this correspondence can be given based on the proof of Theorem 3.

VI. NONUNIFORM FILTER BANKS

In this section we study using the general structures in nonuniform filter banks to achieve what are otherwise impossible.

Consider the three-channel nonuniform filter bank in Fig. 3, built via traditional blocks. It is shown that this system is *incompatible* [14], and hence alias cancellation is impossible using LTI and causal filters, let alone perfect reconstruction. Let the filter bank system $u \mapsto y$ in Fig. 3 be G. To see the structural dependency constraint, we need to compute the blocked transfer matrix for $\underline{G} := L_6 G L_6^{-1}$. Bring in the 6-fold, type-1 polyphase decompositions [34] for the analysis and synthesis filters:

$$\hat{H}_j(z) = \sum_{i=0}^{5} z^{-i}\hat{H}_j^i(z^6),\ \hat{F}_j(z) = \sum_{i=0}^{5} z^{-i}\hat{F}_j^i(z^6),$$
$$j = 0, 1, 2.$$

It follows from the procedures in [4] and [5] that $\underline{\hat{G}}(z) = \underline{\hat{F}}(z)\underline{\hat{H}}(z)$, where $\underline{\hat{H}}(z)$ and $\underline{\hat{F}}(z)$ are, respectively, the analysis and synthesis matrices shown in (23) and (24) at the bottom of the page. Note the structural dependency: The first three columns in $\underline{\hat{F}}(z)$ are mutually dependent and so are the next two columns; similarly, the first three rows in $\underline{\hat{H}}(z)$ are mutually dependent and so are the next two rows. The structural dependency poses a difficulty in design [5].

Shenoy [30] first observed that structural dependency disappears if more general structures are used. Now we propose to use general dual-rate systems as in Fig. 4 to replace the analysis and synthesis subsystems. Note the way Fig. 4 is constructed: each channel has the same decimation ratio as the corresponding channel in Fig. 3, but all three analysis subsystems have the same n, the least common multiple of the three decimation integers $\{2, 3, 6\}$ in Fig. 3; this way, H_0, H_1, F_0, and F_1 in Fig. 4 are more general because they have common factors in their m and n. Denote the system $u \mapsto y$ in Fig. 4 by T. It is easily verified that T is LPTV with period 6; the equivalent blocked system ($\underline{T} = L_6 T L_6^{-1}$) is

$$\begin{aligned}\underline{T} &= L_6[F_0 \quad F_1 \quad F_2]\begin{bmatrix} H_0 \\ H_1 \\ H_2 \end{bmatrix} L_6^{-1} \\ &= [L_6F_0L_3^{-1} \quad L_6F_1L_2^{-1} \quad L_6F_2]\begin{bmatrix} L_3H_0L_6^{-1} \\ L_2H_1L_6^{-1} \\ H_2L_6^{-1} \end{bmatrix} \\ &=: [\underline{F}_0 \quad \underline{F}_1 \quad \underline{F}_2]\begin{bmatrix} \underline{H}_0 \\ \underline{H}_1 \\ \underline{H}_2 \end{bmatrix} \\ &=: \underline{F}\,\underline{H}\end{aligned}$$

$$\underline{\hat{H}}(z) = \begin{bmatrix} \hat{H}_0^0(z) & z^{-1}\hat{H}_0^5(z) & z^{-1}\hat{H}_0^4(z) & z^{-1}\hat{H}_0^3(z) & z^{-1}\hat{H}_0^2(z) & z^{-1}\hat{H}_0^1(z) \\ \hat{H}_0^2(z) & \hat{H}_0^1(z) & \hat{H}_0^0(z) & z^{-1}\hat{H}_0^5(z) & z^{-1}\hat{H}_0^4(z) & z^{-1}\hat{H}_0^3(z) \\ \hat{H}_0^4(z) & \hat{H}_0^3(z) & \hat{H}_0^2(z) & \hat{H}_0^1(z) & \hat{H}_0^0(z) & z^{-1}\hat{H}_0^5(z) \\ \hat{H}_1^0(z) & z^{-1}\hat{H}_1^5(z) & z^{-1}\hat{H}_1^4(z) & z^{-1}\hat{H}_1^3(z) & z^{-1}\hat{H}_1^2(z) & z^{-1}\hat{H}_1^1(z) \\ \hat{H}_1^3(z) & \hat{H}_1^2(z) & \hat{H}_1^1(z) & \hat{H}_1^0(z) & z^{-1}\hat{H}_1^5(z) & z^{-1}\hat{H}_1^4(z) \\ \hat{H}_2^0(z) & z^{-1}\hat{H}_2^5(z) & z^{-1}\hat{H}_2^4(z) & z^{-1}\hat{H}_2^3(z) & z^{-1}\hat{H}_2^2(z) & z^{-1}\hat{H}_2^1(z) \end{bmatrix} \tag{23}$$

$$\underline{\hat{F}}(z) = \begin{bmatrix} \hat{F}_0^0(z) & z^{-1}\hat{F}_0^4(z) & z^{-1}\hat{F}_0^2(z) & \hat{F}_1^0(z) & z^{-1}\hat{F}_1^3(z) & \hat{F}_2^0(z) \\ \hat{F}_0^1(z) & z^{-1}\hat{F}_0^5(z) & z^{-1}\hat{F}_0^3(z) & \hat{F}_1^1(z) & z^{-1}\hat{F}_1^4(z) & \hat{F}_2^1(z) \\ \hat{F}_0^2(z) & \hat{F}_0^0(z) & z^{-1}\hat{F}_0^4(z) & \hat{F}_1^2(z) & z^{-1}\hat{F}_1^5(z) & \hat{F}_2^2(z) \\ \hat{F}_0^3(z) & \hat{F}_0^1(z) & z^{-1}\hat{F}_0^5(z) & \hat{F}_1^3(z) & \hat{F}_1^0(z) & \hat{F}_2^3(z) \\ \hat{F}_0^4(z) & \hat{F}_0^2(z) & \hat{F}_0^0(z) & \hat{F}_1^4(z) & \hat{F}_1^1(z) & \hat{F}_2^4(z) \\ \hat{F}_0^5(z) & \hat{F}_0^3(z) & \hat{F}_0^1(z) & \hat{F}_1^5(z) & \hat{F}_1^2(z) & \hat{F}_2^5(z) \end{bmatrix} \tag{24}$$

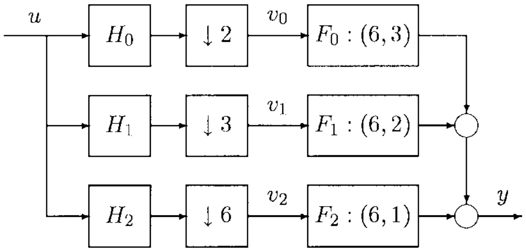

Fig. 12. A filter bank with mixed structures.

where $\underline{H}$ and $\underline{F}$ are both LTI and 6 × 6: $\underline{\hat{H}}(z)$ is the analysis matrix, and $\underline{\hat{F}}(z)$ the synthesis matrix. Note that all entries in $\underline{\hat{H}}$ and $\underline{\hat{F}}$ are freely designable (no structural dependency). Hence, T can perfectly match any causal, LPTV system with period 6; in particular, it can perfectly match any time-delay system by proper choice of the dual-rate systems. Therefore, perfect reconstruction for the structure in Fig. 4 is *possible.*

It is worth noting that if dual-rate structures are used to build nonuniform filter banks as suggested in Fig. 4, the blocked filter-bank systems have the form $\underline{\hat{F}}(z)\underline{\hat{H}}(z)$, where no structural constraints exist among the elements in the two matrices $\underline{\hat{F}}(z)$ and $\underline{\hat{H}}(z)$. This situation is very similar to the polyphase matrix representation of uniform filter banks. Hence, design techniques for uniform filter banks are immediately applicable. A more interesting scenario is perhaps given in Fig. 12, in which the analysis structure in Fig. 3 is combined with the synthesis structure in Fig. 4. Is perfect reconstruction possible in this case?

The answer is positive and is illustrated by the simple example below. (Note that it suffices to construct an analysis filter bank so that the corresponding analysis matrix $\underline{\hat{H}}$ has no finite unstable zeros; then we just take the synthesis matrix $\underline{\hat{F}}$ to be $\underline{\hat{H}}^{-1}$ multiplying some time delay.) Let the LTI analysis filters in Fig. 12 be

$$\hat{H}_0(z) = 1,\ \hat{H}_1(z) = z^{-4} + z^{-5},\ \hat{H}_2(z) = z^{-3}.$$

The associated analysis matrix can be computed from (23):

$$\underline{\hat{H}}(z) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & z^{-1} & z^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z^{-1} & z^{-1} \\ 0 & 0 & 0 & z^{-1} & 0 & 0 \end{bmatrix}.$$

Define the synthesis matrix to be

$$\underline{\hat{F}}(z) = \begin{bmatrix} z^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & -z^{-1} & 0 & 1 & 0 & 0 \\ 0 & z^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & z^{-1} & 0 & 0 & 0 \\ 0 & 0 & -z^{-1} & 0 & 1 & 0 \end{bmatrix}. \tag{25}$$

It follows that $\underline{\hat{F}}(z)\underline{\hat{H}}(z) = z^{-1}I$, and hence the system achieves perfect reconstruction with time delay z^{-6}. The synthesis matrix in (25) is not subject to the structural dependency constraint as in (24) and hence does not give rise to the traditional synthesis structure in Fig. 3; it corresponds to some general dual-rate structures in Fig. 12.

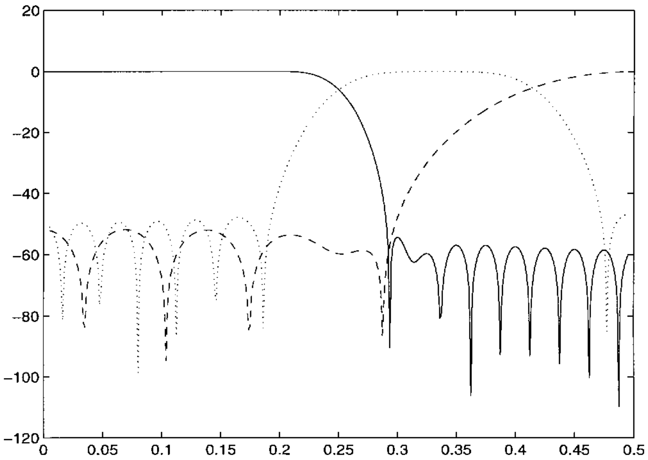

Fig. 13. $|\hat{H}_0|$ (solid), $|\hat{H}_1|$ (dot), and $|\hat{H}_2|$ (dash) in decibels versus $\omega/2\pi$.

The above observation can be generalized to nonuniform multirate filter banks with multiple channels and arbitrary fractional decimation ratios [27], [18], [5]:

> If the synthesis subsystems are replaced by appropriate dual-rate structures, incompatibility [14] and structural dependency [18], [5] disappear; and perfect reconstruction is always possible.

One advantage of eliminating structural dependency is to allow optimal design of synthesis systems; we conclude this section by considering such a design example. We use the structure in Fig. 12 and preselect the linear-phase, FIR analysis filters: H_0 (order 40) is lowpass with cutoff frequency $\omega = \pi/2$; H_1 (order 30) is bandpass with passband $\pi/2 \leq \omega \leq 5\pi/6$; H_2 (order 14) is highpass with cutoff frequency $5\pi/6$. All three filters are designed using MATLAB function $fir1$ with their magnitude Bode plots given in Fig. 13.

The synthesis systems now can be designed by minimizing the ℓ_2-induced norm [6] of the error system between an ideal time-delay system T_d and the filter bank T:

$$J_{opt} := \min_{F_i} \|T_d - T\|.$$

By blocking, we can reduce this optimization problem to an equivalent $\mathcal{H}_\infty$ model-matching problem [6], [5]

$$J_{opt} = \min_{\underline{\hat{F}}(z)} \|\underline{\hat{T}}_d(z) - \underline{\hat{F}}(z)\underline{\hat{H}}(z)\|_\infty$$

where $\underline{\hat{T}}_d(z)$ is the 6-fold blocked time-delay system, $\underline{\hat{H}}(z)$ the associated 6 × 6 analysis matrix given in (23), $\underline{\hat{F}}(z)$ the associated 6 × 6 synthesis matrix with no structural dependency, and the norm is the $\mathcal{H}_\infty$ norm [12], [13] (the peak maximum singular value of the frequency-response matrix). The later optimization problem is solvable [12], [13] by, e.g., the MATLAB software—μ-Analysis and Synthesis Toolbox [2].

For good reconstruction error (J_{opt}), we take the ideal time delay to be $\hat{T}_d(z) = z^{-80}$. The $\mathcal{H}_\infty$ optimization then yields a reconstruction error $J_{opt} = 0.31\%$. This means [23] that the maximum alias and magnitude distortions, defined appropriately as in [23], of the designed system are both ≤0.31% and the phase distortion is $\leq \sin^{-1}$ (0.31%) = 0.18°. Of course, the synthesis systems designed use the general dual-rate structures.

VII. CONCLUSION

In this paper we proposed dual-rate systems as general building blocks for multirate signal processing. Several structures implementing such dual-rate systems were studied in detail. The work in this paper provides a unified framework for using several generalized elements such as block decimation and expansion, LPTV filtering, and vector sample-rate changers, which have already found applications in multirate filter banks [17], [26], [38]. The advantages of such general dual-rate systems in nonuniform filter banks were illustrated: Channel incompatibility [14] and structural dependency [18], [5] are removable by proper use of the dual-rate structures.

The structures studied in this paper fall into the general class of LPTV systems. This fact is essential for using the blocking technique to relate to equivalent single-rate, LTI systems which are necessarily multivariable. The techniques in this paper are not directly applicable to more general structures which are not periodically time-varying; examples of such systems include the linear time-varying filter banks studied in [32] and the references therein. For the time-varying multirate systems, time-domain techniques are proven to be more effective.

ACKNOWLEDGMENT

The authors are grateful to Dr. M. Khansari and Dr. R. Shenoy for helpful discussions, and especially to Dr. Shenoy for an advance copy of [30]. They would also like to thank the anonymous reviewers for their helpful comments/suggestions.

REFERENCES

[1] A. N. Akansu and R. A. Haddad, *Multiresolution Signal Decomposition.* New York: Academic, 1992.
[2] G. J. Balas, J. C. Doyle, K. Glover, A. Packard, and R. Smith, "μ-Analysis and synthesis toolbox user's guide," MuSyn Inc. and the MathWorks, Inc., 1991.
[3] C. W. Barnes and S. Shinnaka, "Block-shift invariance and block implementation of discrete-time filters," *IEEE Trans. Circuits Syst.,* vol. CAS-27, pp. 667–672, 1980.
[4] T. Chen, "A lifting approach to analysis and design of nonuniform multirate filter banks," in *Proc. Canadian Conf. Electrical Computer Eng.,* Montreal, P.Q., Canada, Sept. 1995, pp. 874–877.
[5] ——, "Nonuniform multirate filter banks: Analysis and design with an $\mathcal{H}_\infty$ performance measure," *IEEE Trans. Signal Processing,* vol. 35, pp. 572–582, 1997.
[6] T. Chen and B. A. Francis, "Design of multirate filter banks by $\mathcal{H}_\infty$ optimization," *IEEE Trans. Signal Processing,* vol. 43, pp. 2822–2830, 1995.
[7] T. Chen and L. Qiu, "$\mathcal{H}_\infty$ design of general multirate sampled-data control systems," *Automatica,* vol. 30, no. 7, pp. 1139–1152, 1994.
[8] T. Chen, L. Qiu, and E. Bai, "General multirate building blocks and their application in nonuniform filter banks," in *Proc. IEEE Int. Symp. Circuits Syst.,* Hong Kong, June 1997, pp. 2349–2352.
[9] R. E. Crochiere and L. R. Rabiner, *Multirate Digital Signal Processing.* Englewood Cliffs, NJ: Prentice-Hall, 1983.
[10] N. J. Fliege, *Multirate Digital Signal Processing.* New York: Wiley, 1994.
[11] G. D. Forney, Jr., "Convolutional codes I: Algebraic structure," *IEEE Trans. Inform. Theory,* vol. 16, pp. 720–738, 1970.
[12] B. A. Francis, *A Course in $\mathcal{H}_\infty$ Control Theory.* Berlin, Germany: Springer-Verlag, Lecture Notes in Control and Information Sciences, 1987, vol. 88.
[13] M. Green and D. J. N. Limebeer, *Linear Robust Control.* Englewood Cliffs, NJ: Prentice-Hall, 1995.
[14] P.-Q. Hoang and P. P. Vaidyanathan, "Non-uniform multirate filter banks: Theory and design," in *Proc. IEEE Int. Symp. Circuits Syst.,* Portland, OR, 1989, pp. 371–374.
[15] C.-C. Hsiao, "Polyphase filter matrix for rational sampling rate conversions," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing,* Dallas, TX, Apr. 1987, pp. 2173–2176.
[16] M. R. K. Khansari and A. Leon-Garcia, "Perfect reconstruction conditions for subband decomposition with generalized subsampling," presented at the *SPIE—Vis. Commun. Image Processing Conf.,* Boston, MA, 1992.
[17] "Subband decomposition of signals with generalized sampling," *IEEE Trans. Signal Processing,* vol. 41, pp. 3365–3376, 1993.
[18] J. Kovačević and M. Vetterli, "Perfect reconstruction filter banks with rational sampling factors," *IEEE Trans. Signal Processing,* vol. 41, pp. 2047–2064, 1993.
[19] H. S. Malvar, *Signal Processing with Lapped Transforms.* Norwood, MA: Artech House, 1992.
[20] D. G. Meyer, "A new class of shift-varying operators, their shift-invariant equivalents, and multirate digital systems," *IEEE Trans. Automat. Contr.,* vol. 35, pp. 429–433, 1990.
[21] R. A. Meyer and C. S. Burrus, "A unified analysis of multirate and periodically time-varying digital filters," *IEEE Trans. Circuits Syst.,* vol. CAS-22, pp. 162–168, 1975.
[22] Y. Meyer, *Wavelets: Algorithms and Applications.* Philadelphia, PA: SIAM, 1993.
[23] S. Mirabbasi, B. A. Francis, and T. Chen, "Controlling distortions in maximally decimated filter banks," *IEEE Trans. Circuits Syst. II,* vol. 44, pp. 597–600, 1997.
[24] S. K. Mitra and R. Gnanasekaran, "Block implementation of recursive digital filters: New structures and properties," *IEEE Trans. Circuits Syst.,* vol. CAS-25, pp. 200–207, 1978.
[25] T. Miyawaki and C. W. Barnes, "Multirate recursive digital filters—A general approach and block structures," *IEEE Trans. Acoust., Speech, Signal Processing,* vol. ASSP-31, pp. 1148–1154, 1983.
[26] K. Nayebi, T. P. Barnwell, III, and M. J. T. Smith, "Block decimated analysis-synthesis filter banks," in *Proc. ISCAS,* 1992, pp. 947–950.
[27] ——, "Nonuniform filter banks: A reconstruction and design theory," *IEEE Trans. Signal Processing,* vol. 41, pp. 1114–1127, 1993.
[28] R. L. Reng, "Polyphase and modulation descriptions of multirate systems—A systematic approach," in *Proc. Int. Conf. DSP,* 1995, pp. 212–217.
[29] R. G. Shenoy, "Analysis of multirate components and application to multirate filter design," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing,* 1994, vol. 3, pp. 121–124.
[30] ——, "Multirate specifications via alias-component matrices," *IEEE Circuits Syst. II,* vol. 45, pp. 314–320, 1998.
[31] R. G. Shenoy, D. Burnside, and T. W. Parks, "Linear periodic systems and multirate filter design," *IEEE Trans. Signal Processing,* vol. 42, pp. 2242–2256, 1994.
[32] I. Sodagar, K. Nayebi, T. P. Barnwell, III, and M. J. T. Smith, "Time-varying analysis-synthesis systems based on filter banks and post filtering," *IEEE Trans. Signal Processing,* vol. 43, pp. 2512–2524, 1995.
[33] G. Strang and T. Q. Nguyen, *Wavelets and Filter Banks.* Wellesley, MA: Wellesley-Cambridge Press, 1996.
[34] P. P. Vaidyanathan, *Multirate Systems and Filter Banks.* Englewood Cliffs, NJ: Prentice-Hall, 1993.
[35] P. P. Vaidyanathan and S. K. Mitra, "Polyphase networks, block digital filtering, LPTV systems, and alias-free QMF banks: A unified approach based on pseudocirculants," *IEEE Trans. Acoust., Speech, Signal Processing,* vol. 36, pp. 381–391, 1988.
[36] M. Vetterli and J. Kovačević, *Wavelets and Subband Coding.* Englewood Cliffs, NJ: Prentice-Hall, 1995.
[37] M. Vetterli and C. Herley, "Wavelets and filter banks: Theory and design," *IEEE Trans. Acoust., Speech, Signal Processing,* vol. 40, pp. 2207–2232, 1992.

[38] X.-G. Xia and B. W. Suter, “Multirate filter banks with block sampling,” *IEEE Trans. Signal Processing,* vol. 44, pp. 484–496, 1996.

[39] ——, “Vector-valued wavelets and vector filter banks,” *IEEE Trans. Signal Processing,* vol. 44, pp. 508–518, 1996.

Tongwen Chen (S’86–M’91–SM’97) received the B.Sc. degree from Tsinghua University, Beijing, China, in 1984, and the M.A.Sc. and Ph.D. degrees from the University of Toronto in 1988 and 1991, respectively, all in electrical engineering.

From October 1991 to April 1997, he was on faculty in the Department of Electrical and Computer Engineering at the University of Calgary, Calgary, Alta., Canada. Since May 1997, he has been with the Department of Electrical and Computer Engineering at the University of Alberta, Edmonton, Alta., and is presently an Associate Professor. His current research interests include digital control, digital signal processing, optimal and robust design, involving especially multirate systems. He co-authored, with B. A. Francis, the book *Optimal Sampled-Data Control Systems* (New York: Springer, 1995).

Dr. Chen is an Associate Editor for IEEE TRANSACTIONS ON AUTOMATIC CONTROL. He is a registered Professional Engineer in Alberta, Canada.

Li Qiu (M’90) received the B.Eng. degree in electrical engineering from Hunan University, Changsha, Hunan, China, in 1981, and the M.A.Sc. and Ph.D. degrees in electrical engineering from the University of Toronto, Toronto, Ont., Canada, in 1987 and 1990, respectively.

Since 1990, he has held research and teaching positions in the University of Toronto, Canadian Space Agency, University of Waterloo, and University of Minnesota. At present, he is an Assistant Professor at the Department of Electrical and Electronic Engineering, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong. His current research interests include robust control, digital control, signal processing, and motor control.

Dr. Qiu is an Associate Editor of the IEEE TRANSACTIONS ON AUTOMATIC CONTROL.

Er-Wei Bai received his education from Fudan University, Shanghai Jiaotong University, and University of California at Berkeley.

He is an Associate Professor of Electrical Engineering, University of Iowa, Iowa City, where he teaches and conducts research in the area of systems, controls, and signal processing.