Truncation Length for Viterbi Decoding
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Abstract—A bound is derived and analyzed for the bit error rate (BER) of a Viterbi decoder with survivor truncation. Estimates of the SNR loss on the AWGN channel, due to truncation, are obtained for convolutional codes. Larger truncation lengths are required than the smallest value that does not effectively decrease the code's free distance, especially at low $E_b/N_0$.

I. INTRODUCTION

In order to design efficient and high-performance Viterbi decoders, one must know the precise loss from maximum likelihood decoding (MLD) when survivor paths are truncated to $T$ trellis branches. The problem is to determine the least value of $T$ such that this "truncation loss" is acceptable (i.e., 0.05 dB on the AWGN channel). Binary rate $k/n$ convolutional codes with total encoder memory $m$ are considered here. The state of an encoder is $s = \sum_{i=1}^{m} s_i 2^{i-1}$ when $s_i$ is contained in the $i$th memory cell. Define a trellis level as the set of metrics, at some time, for all $2^m$ possible encoder states. For each trellis level, a best state, truncation length $T$, Viterbi decoder outputs the information bit(s) for the $T$th previous branch of the survivor path into the state having lowest metric. When it is expensive to find this state, a fixed state decoder, which always outputs bits from the survivor into one fixed state, is preferred. However, bounding the resulting BER is difficult, so $T$ is estimated as roughly twice the value computed for a best state decoder [3]. For those (practical) decoders which output bits for several trellis branches at a time, $T$ should be interpreted as the average truncation length because the average BER over the truncation lengths used is not significantly different from the BER of a decoder using the average truncation length to decode one branch at a time. The exact choice of $T$ depends upon the code, channel noise level, acceptable loss from MLD, and the manner in which the Viterbi decoder outputs bits. For a description of survivor memory management and traceback, the reader should consult [4].

The BER for a best state, truncation $T$, Viterbi decoder on a binary-input, output-symmetric, memoryless channel is upper-bounded in the next section as

$$P_b^{(T)} \leq \sum_{d=0}^{\infty} \left[ b(d) + \frac{1}{2^{k-1}} \sum_{s=1}^{2^m-1} a_s(d,T) - \frac{1}{2} a_s(d,T+1) \right] P_d$$

(1)

where $P_d$ is the probability that the decoder chooses a path at distance $d$ from the one transmitted and $b(d)$ is the total weight of all information sequences producing weight $d$ trellis paths which go from state 0 to state 0 via nonzero states. Also, $a_s(d,T)$ is the number of weight $d$ trellis paths, having length $T$ or more branches, which go from state 0 into state $s$ via nonzero states. For trellis codes, $b(d)$ and $a_s(d,T)$ in (1) are averages, over all possible trellis paths sent, for paths at squared Euclidean distance $d$ from the one transmitted [7].

For rate $(n-1)/n$ punctured codes,

$$P_b^{(T)} \leq \frac{1}{n-1} \sum_{d=0}^{\infty} \left[ b(d) + \sum_{i=1}^{n-1} \sum_{s=1}^{2^m-1} a_s^{(i)}(d,T) - \frac{1}{2} a_s^{(i)}(d,T+1) \right] P_d$$

(2)

because $T$ is measured, not in branches for $k$ input bits, but in trellis branches for the underlying rate $1/2$ code that the Viterbi algorithm operates on when decoding the punctured code. Also, $a_s^{(i)}(d,T)$ is replaced by an average over the $n-1$ possible trellises at any instance of time ($i$ indexes them).

These bounds generalize and strengthen a previous one [1]. Simulations of the memory 6, rate 1/2 NASA standard code at a BER of $10^{-5}$ on the unquantized AWGN channel show that a Viterbi decoder with $T = 27, 30$ or $T = \infty$ (i.e., MLD) performs within 0.1 dB of the $P_b^{(T)}$ bound (1). For other codes, similar accuracy is expected when the MLD bound $1/k \sum b(d) P_d$ is tight.

In the next section, best state decoding with survivor truncation length $T$ is analyzed. In Section III, an algorithm is described for computing the first few (say 30) nonzero terms in the infinite series (1) or (2). Last, these results are applied to several convolutional codes.
II. BEST STATE TRUNCATED VITERBI DECODING

When computing error probabilities for an output-symmetric channel, one may assume that the encoder transmits all zeros [6, p. 87]. At level \( j + T \), let \( \ell \) denote the survivor path into the state \( s \) that has lowest metric. Consider decoding bit(s) for the trellis branch from level \( j \) to \( j + 1 \). Decoder bit error(s) occur i) if \( \ell \) merges with state 0 at some level \( > j \) and nonzero information bit(s) correspond to the branch from level \( j \) to \( j + 1 \), or else ii) if \( \ell \) merges with state 0 at level \( j \) but not afterwards, or else iii) with probability 1/2 (on average) if \( \ell \) merges with state 0 at level \( < j \) but not afterwards (see Fig. 1). First, note that MLD errors overbound those in case i). With probability \( P_d \), the decoder selects one of the \( a_i(d, T + 1) \) weight \( d \) trellis paths of type iii). One of the \( a_i(d, T) - a_i(d, T + 1) \) weight \( d \) trellis paths of type ii) causes an average of \( k2^{k-1}/(2^k - 1) \) bit errors when \( k \) bits are decoded for the branch leaving state 0. Applying a union bound yields (1). For rate \( (n - 1)/n \) punctured codes derived from rate 1/2 convolutional codes [5], error probabilities resulting from the above situations must be averaged over the \( n - 1 \) possible trellises when the branch from level \( j \) to \( j + 1 \) is decoded. Thus, (2) results from replacing \( a_i(d, T) \) in (1) by an average value for \( n - 1 \) trellises.

For each state in a rate 1/\( n \) decoder, the newest survivor bit is the oldest bit (sm) of the predecessor state whose metric plus \( n \) physical storage bits \( (kT - m \) in general) per survivor are required for truncation \( T \) decoding.

Define \( T^* \) as the least value of \( T \) such that \( a_i(d, T) = 0 \) for all \( d \leq d_{occ} \); for punctured codes, \( a_i^{(1)}(d, T^*) = 0 \) for \( i = 1, \ldots, n - 1 \) and \( d \leq d_{occ} \). A decoder with truncation length \( T^* \) will have no loss from MLD on "asymptotically quiet" channels such as the AWGN with extremely high bit signal-to-noise ratio \( E_b/N_0 \). In [1] and [2], decoding with truncation \( T^* \) is recommended but this may cause a large loss, for example, in \( E_b/N_0 \) on the AWGN channel, at decoded bit error rates greater than \( 10^{-6} \).

\( T^* \) values are listed in Table I for several convolutional codes. Each code is referenced by generator polynomials, for a feedforward encoder, shown in octal form and right justified as in [5] so that \( 1 + x^2 + x^3 \) appears as 13. The code which has an encoder with generators \( g_1(x) = 1 + x^2 + x^3 \) and \( g_2(x) = 1 + x + x^2 + x^3 \) will be called the (13,17) code herein. The fast growth of path distances with length for the two systematic codes results in relatively small values of \( T^* \).

The performance of a maximum-likelihood decoder on a memoryless channel does not change when all encoder generator polynomials are reversed. For if an information vector \( i \), starting and ending with \( m \) zeros, encodes to \( c \), then \( c \) reversed is the output of the reversed encoder with input \( i \) reversed. However, if any generator polynomial is not symmetric, the truncation loss for a Viterbi decoder (with \( T \) moderate) may change because the truncated trellis branch labels and thus \( a_i(d, T) \) values are different. Therefore, encoders should be designed with generator polynomials oriented to minimize truncation loss.

Example: For the \((15,17)\) code, \( P_b(10) \leq 4P_b + 32P_b + 102P_b + 254P_b \cdots \) while for the reversed code \((13,17)\), \( P_b(10) \leq 2P_b + 20P_b + 85P_b + 223P_b + \cdots \). (The MLD BER \( \leq 7P_b + 7P_b + 18P_b + 49P_b + \cdots \).) On the unquantized AWGN channel at \( E_b/N_0 = 5.41 \) dB, the above bounds are \( 3.42 \times 10^{-5}, 2.66 \times 10^{-5}, \) and \( 10^{-5}\), respectively. (The first 20 nonzero terms in \( P_b \) were sufficient for 3 significant digits of precision.) An additional 0.40 or 0.32 dB is required when \( T_b = 10 \) for the \((15,17)\) or \((13,17)\) code, respectively, to achieve a BER of \( 10^{-5}\). These losses are excessive because only 0.05 dB extra is needed when \( T_b = 14 \). Furthermore, \( P_b(10) \leq 5 \times 10^{-5} \) for the \((5,7)\) code at \( E_b/N_0 = 5.94 \) dB, with only half the decoding computation. Hence, truncation length \( T^* = 10 \) for the \((15,17)\) code is insufficient, unless the BER is \( < 10^{-8}\). For the unquantized AWGN channel, \( P_b = Q \left( \sqrt{2(E_b/N_0)/L} \right) \) where \( Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-y^2/2} dy \). In order to calculate the bounds on \( P_b(T) \), the bound \( Q(x) < \frac{e^{-x^2/2}}{x} \) for \( x > 0 \) was used because it has a relative error \( < 0.004 \) when \( Q(x) < 10^{-4}\).

III. AN ALGORITHM FOR COUNTING PATHS AND BITS

Viterbi’s algorithm, with survivors replaced by vectors of integers that count paths or bits, is used to compute \( b(d) \) and \( a_i(d, T) \) values. Rate \( 1/n \) codes are treated first to simplify the discussion. Define \( \text{out}_{b}[s] \) and \( \text{out}_{i}[s] \) as the number of ones that the encoder outputs when entering state \( s \) from state...
the least weight of any simple path from state 0 to N[s][0] or N[s][d - N[s][0] + 1] = \alpha(s, d, 1)

The algorithm presented is certainly simple but not efficient. The double-buffering matrices P and A may be eliminated with in-place computation of all matrix values [4]. Replacing the terminating condition change by a flag for each value of t would reduce the work done (but increase the complexity).

For uncoded 1's output by the encoder entering state s, with binary input j ∈ {0, 1, 2k-1}, N[s][t] and B[s][t] are computed using 2k entries from each of P and A. In Tables I and II, T0 is the least value of T such that a Viterbi decoder will perform within 0.05 dB of ML performance on the quantized AWGN channel, according to the bound (1) or (2), for BER ≲ 10^{-5}; E_b/N_0 is the required bit signal-to-noise ratio. In Tables I and II, change became nonzero after 100 steps. The algorithm requires storage for 4 \cdot 2^{m+2} \cdot (m + 2) integers and the amount of work per step is proportional to this number. After step T* (level j + 1 in the trellis), P[0][0] is set to 999 in order to inhibit paths from state 0. Also, A, B and change may be ignored. Then for the Tth next step, \alpha(d, T) = N[d - N[s][0] + 1] for d = N[s][0] to N[s][d] terms - 1.

Truncating the sum on d in (1) or (2) at d = d_{free} + \lceil 10n/k \rceil yields results with 2 or 3 digits of precision when the bounds are exact enough to be useful. Thus, terms may be set to \lceil 10n/k \rceil. Since exact b(d) or \alpha(d, T) values are unnecessary for d >> d_{free}, terms could be decreased because ratios of successive nonzero coefficients would generate additional (although approximate) values.
Since each survivor is \( L_b = T_b - m \) bits long, the decoder's path memory grows linearly with \( T \). However, the decoder complexity doubles for each increase in \( m \) by one, while the \( E_b/N_0 \) needed for a BER \( \leq 10^{-4} \) decreases less than 0.4 dB for most codes shown. Therefore, a truncation length of at least \( T_b \) should be used. A larger \( T_b \) may be needed as \( m \) increases beyond 6 because the \( E_b/N_0 \) gain decreases so the acceptable truncation loss is reduced. For higher/lower bit error rates, \( T_b \) must be greater/smaller in order to maintain a fixed \( E_b/N_0 \) loss. For decoders which output a block of bits at a time, the average truncation length should be at least \( T_b \). For fixed state Viterbi decoders, survivors should be \( 2L_b \) bits long [3].

The \( T_b = 5K = 5(m + 1) \) "five constraint lengths rule," for rate 1/2 nonsystematic convolutional codes with \( m < 10 \), matches the value in Table I for best state decoding of the (133,171) code, but differs markedly for the other codes. As \( m \) increases beyond 6, \( T_b \) is much bigger than \( T_b^* \) because a large number of states causes \( \sum a_s(d,T) \) to grow more rapidly with \( d \) than the \( b(d) \) sequence, so the bound (1) becomes loose at a BER of \( 10^{-3} \). According to simulations on an 8-bit quantized AWGN channel, \( T_b = 45 \) instead of 53 is sufficient for the (1765,1631,1327) code at BER \( = 10^{-3} \) (but \( T_b = 64 \) is required at BER \( = 10^{-3} \) and \( T_b = 36 \) at BER \( = 10^{-6} \)). When a convolutional code is concatenated with an outer block code, the Viterbi decoder usually operates at a BER near \( 10^{-3} \), in which case \( T \) must be larger than the value for a BER of \( 10^{-5} \). According to simulations, \( T_b = 40 \) is needed for the (133,171) NASA code and \( T_b = 34, 38, \) or 56 are required for the (75,57) code, (133,171,165) code, and (133,171,133,0) punctured code, respectively, all at a BER of \( 10^{-3} \).

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REFERENCES