Total Energy Shaping Control

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Objectives

– Present a systematic procedure to incorporate *energy information* in control design.
– Clarify the roles of: energy–balancing as a stabilization mechanism, and the *dissipation* obstacle.
– Underscore the generality of the methodology.
– Illustrate the application with physical examples.

Layout

1. Introduction.
2. Passivity and energy–shaping.
3. Stabilization via energy–balancing (EB).
4. Dissipation obstacle.
5. Port–controlled Hamiltonian models.
6. Interconnection and damping assignment control.
7. Examples: Power converters, AC motors and mechanical systems.
8. Current research and open problems.

1. Introduction

A. Inteligent control paradigm revisited

- Control design problems traditionally approached adopting a signal–processing viewpoint.

- Objectives: keep some error signals small and reduce the effect of certain disturbance inputs in spite of unmodeled dynamic.

♡ Discriminated via filtering. Very successful for linear time–invariant (LTI) systems

† Impossible in nonlinear case: i) far from obvious computations, ii) nonlinear systems "mix" the frequencies.

† "Crank–up" the gain to quench the (large set of) undesirable signals...*utmost impractical!*.[a]

How to incorporate prior *structural* information?

⇒ Our inability is inherent to the signal–processing viewpoint.

[a]Intrinsically conservative, amplifies noise, energy consumption...

B. Contents

- To incorporate *energy principles* in control we adopt a control–as–interconnection framework and view dynamical systems (plant and controller) as *energy–transformation* devices interconnected to achieve desired behaviour.

- Consider physical systems, i.e., that satisfy *energy–balancing*. Control problem is to assign a desired energy function.

♡ Advantages of adopting an **energy–shaping** perspective:

1. Aim at, not just stabilization, but also *performance* objectives.

2. Energy is a fundamental concept that can serve as a *lingua franca* to communicate with practitioners, incorporate prior knowledge and provide physical interpretations to the control action.

C. Approaches to energy shaping

- The idea has its roots in robot control *(Takegaki/Arimoto,'81)*. Also *(Jonckheree,'81)*.

♡ Principle formalized in *(Ortega/Spong,'89)*, via definition of *passivity–based control* (**PBC**):

– Control as interconnections of passive systems ⇒ *energy–balancing* interpretation of stabilization;

– Approach hinges upon the fundamental (and universal) property of passivity ⇒ can be *extended* to many applications.

- Two approaches to PBC:

i) Standard: Fix *a priori* the desired storage function (typically quadratic in the increments.) Problems: Not an *energy* function and stabilization mechanism akin to systems inversion.

ii) *Interconnection and damping assignment* *(IDA)*: Storage function –now a *bona fide* energy function– obtained *as a result* of our choice of desired subsystems interconnections and damping.

- **Applications of IDA–PBC**: mass–balance syst., electrical machines, power syst., underwater vehicules, magnetic levitation, underactuated mechanical syst., and power converters.

D. Basic References: Theory

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2. Passivity and energy–shaping

- Lumped parameter systems interconnected to the external environment through *port power conjugated variables* $u \in I\!R^m$ and $y \in I\!R^m$ (product has units of power). They satisfy the **energy–balance equation** (EBE)

$$\underbrace{H[x(t)] - H[x(0)]}_{stored\ \ energy} = \underbrace{\int_0^t u^\top(s)y(s)ds}_{supplied} - \underbrace{d(x(t),t)}_{dissipated\ \geq 0}$$

Systems that satisfy EBE with $H(x) \geq c$ are **passive**, and y is called the passive output.

A. Key observations

- With $u \equiv 0$, we have $H[x(t)] \leq H[x(0)]$. Will stop in a point of *minimum energy*.

- Control introduced to operate the system around some non–zero equilibrium point, say x_*.

- Rate of convergence increased if we extract energy $u = -K_{di} y$, with $K_{di} = K_{di}^\top > 0$.

- $-\int_0^t u^\top(s) y(s) ds \leq H[x(0)] < \infty \quad \Leftrightarrow$ amount of energy that can be extracted from a passive system is *bounded.*

- $d(x(t), t)$, as function of t, is non–decreasing; typically $\int_0^t (\cdot)^2 ds$.

B. Examples

- *Electrical circuits*

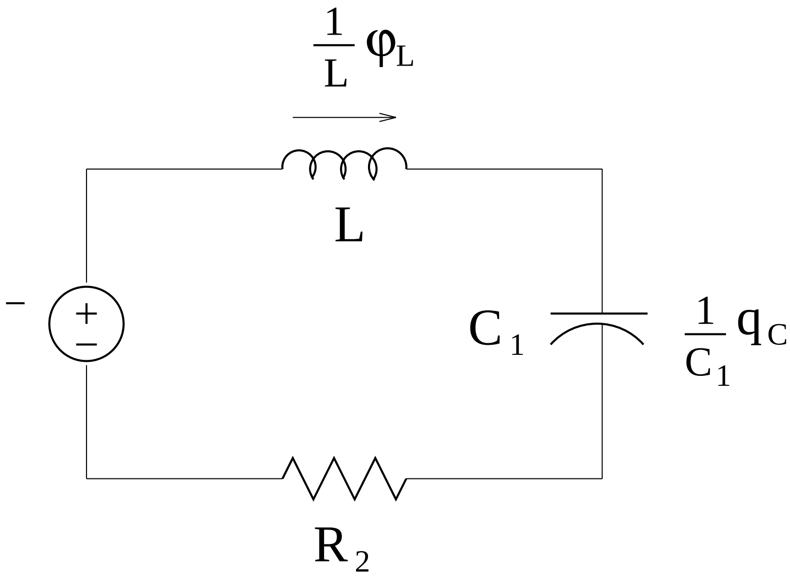

Figure 1:

- Model, with $x \triangleq [q_C, \phi_L]^\top$,

$$\Sigma : \begin{cases} \dot{x}_1 = \frac{1}{L} x_2 \\ \dot{x}_2 = -\frac{1}{C} x_1 - \frac{R}{L} x_2 + u \\ y = \frac{1}{L} x_2 \end{cases}$$

- Energy $H(x) = \frac{1}{2C} x_1^2 + \frac{1}{2L} x_2^2$.

- Satisfies **(EBE)** with $d(x(t), t) = R \int_0^t [\frac{1}{L} x_2(s)]^2 ds$.

- *Mechanical systems*

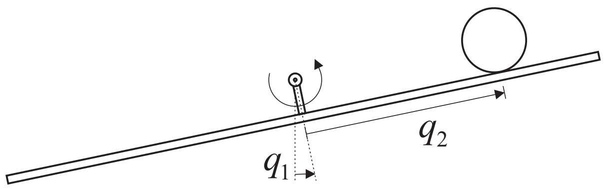

Figure 2:

- Model[a]

$$\begin{aligned} m_2\ddot{q}_1 + m_3\sin(q_2) - m_1 q_1 \dot{q}_2^2 &= 0 \\ (1+m_1 q_1^2)\ddot{q}_2 + 2m_1 q_1 \dot{q}_1 \dot{q}_2 + m_3 q_1 \cos(q_2) &= u \end{aligned}$$

- Energy $H = \frac{1}{2}\dot{q}^\top M(q_1)\dot{q} + m_3 q_1 \sin(q_2)$, where

$$M(q_1) = \begin{bmatrix} m_2 & 0 \\ 0 & 1+m_1 q_1^2 \end{bmatrix}$$

- Satisfies **(EBE)** with $d(x(t),t) = 0$ and $y = \dot{q}_2$.

[a] Wrong coordinates!

- *Electromechanical systems*

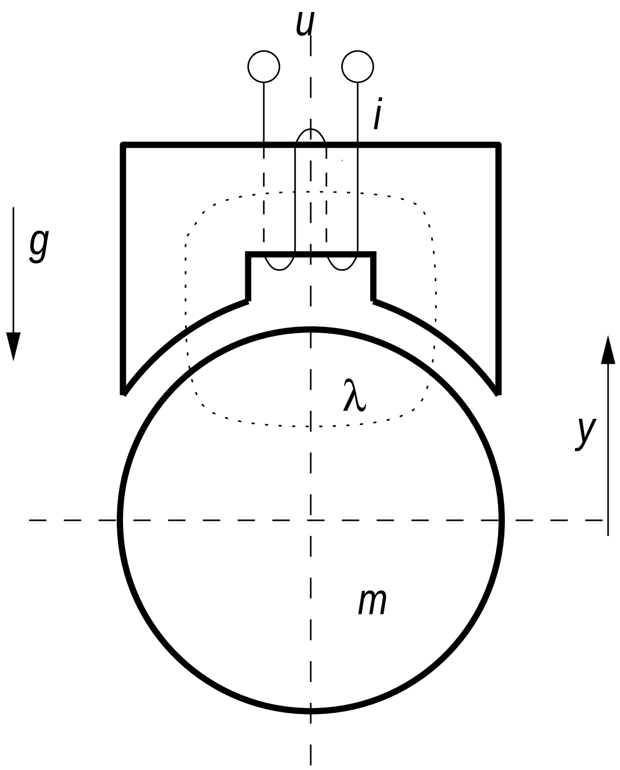

Figure 3:

- Model (Assuming linear magnetics, i.e., $\lambda = L(\theta)i$)

$$\begin{aligned}\dot{\lambda} + Ri &= u \\ m\ddot{\theta} &= F - mg \\ F &= \frac{1}{2}\frac{\partial L}{\partial \theta}(\theta)i^2\end{aligned}$$

- Total energy:

$$H = \frac{1}{2}\frac{\lambda^2}{L(\theta)} + \frac{m}{2}\dot{\theta}^2 + mg\theta$$

- Output $y = i$, same dissipation. Notice, however, that H is *not bounded* from below!

- *Power converters*

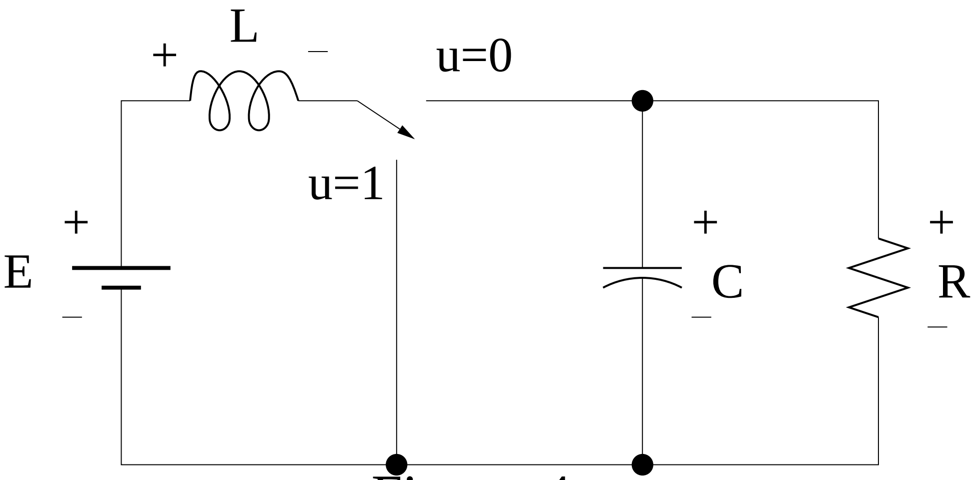

Figure 4:

- Model, with $x \triangleq [\phi_L, q_C]^\top$,

$$\begin{aligned}\dot{x}_1 &= -u\frac{1}{C}x_2 + E \\ \dot{x}_2 &= u\frac{1}{L}x_1 - \frac{1}{RC}x_2\end{aligned}$$

- Total energy: $H(x) = \frac{1}{2L}x_1^2 + \frac{1}{2C}x_2^2$.
- Attention: “Input” and output: $E \mapsto \frac{x_1}{L}$!

C. Standard formulation of PBC

Select a control action[a] $u = \beta(x) + v$ so that

$$H_d[x(t)] - H_d[x(0)] = \int_0^t v^\top(s) z(s) ds - d_d(x,t)$$

where

- $H_d(x)$, the desired total energy function, has a *minimum* at x_*,
- $d_d(x,t) \geq 0$ desired damping, and
- z (which may be equal to y) is the new passive output

$\Leftrightarrow$

energy–shaping plus damping injection.

[a] State feedback, for ease of presentation

D. Discussion

– Labeling inputs u and outputs y is *restrictive*, and the "control–as–interconnection" perspective is needed to cover a wider range of applications.

– u may contain some external variables like disturbances or sources. Control may not enter at all in u! (e.g., converter example)

– The choice of the desired damping is *far from obvious* and maybe deleterious for *performance*.

– Automatically ensures some *robust stability* (e.g., frictions and parasitic resistances).

– Passivity can be used for stabilization *independently* of energy–shaping, finding a "detectable" $z = h(x)$.

E. Connections with $\mathcal{L}_2$–gain assignment

- **Fact** If the dissipation is such that

$$d_d(t) \geq \delta \int_0^t |z(s)|^2 ds$$

for some $\delta > 0$, then the map $v \mapsto z$ has $\mathcal{L}_2$ gain smaller than $\frac{1}{\delta}$.

Proof From the new (EBE) and above

$$\int_0^t v^\top(s) z(s) ds \geq \delta \int_0^t |z(s)|^2 ds - H_d[x(0)]$$

which is equivalent to

$$\begin{aligned} & \frac{1}{2}\int_0^t |z(s)|^2 ds \leq \\ \leq \quad & -\frac{1}{2}\int_0^t |z(s)|^2 ds + \frac{1}{\delta}\int_0^t v^\top(s) z(s) ds + \frac{H_d[x(0)]}{\delta} \\ \leq \quad & -\frac{1}{2}\int_0^t |z(s) - \frac{1}{\delta} v(s)|^2 ds + \frac{1}{2\delta^2}\int_0^t v^\top(s) z(s) ds + \frac{H_d[x(0)]}{\delta} \end{aligned}$$

Thus

$$\int_0^t |z(s)|^2 ds \leq \frac{1}{\delta^2}\int_0^t |v(s)|^2 ds + \frac{1}{\delta} H_d[x(0)]$$

- Achieved choosing *damping injection*

$$v = K_{di} z + w, \; K_{di} = K_{di}^T \geq \delta I > 0$$

3. Stabilization via energy–balancing

For a class of syst., including *mechanical*, the solution is very simple:

• Find $\beta(x)$ s. t. the energy supplied by the controller is a function of the syst. state.

Indeed, *if*

$$-\int_0^t \beta^\top[x(s)]y(s)ds = H_a[x(t)] + \kappa$$

for some $H_a(x)$, then

$$u = \beta(x) + v$$

ensures $v \mapsto y$ is passive with *new energy function*

$$\boxed{H_d(x) \triangleq \underbrace{H(x)}_{stored} + \underbrace{H_a(x)}_{-\ supplied}} \Leftrightarrow \boxed{\textbf{EB-PBC}}$$

A. Mechanical systems

Lagrangian and Hamiltonian modelling

1) *Energy function* in terms of generalized variables $q \in I\!R^n$ (positions and charges), which leads to the Lagrangian function:

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2}\dot{q}^\top M(q)\dot{q} - V(q)$$

where $M(q) = M^\top(q) > 0$.

2) *Equations of motion* derived invoking, e.g., Hamilton's principle, which (roughly speaking) states that the system moves along trajectories that minimize the integral of the Lagrangian.[a]

$$\frac{d}{dt}[\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}}] - \frac{\partial \mathcal{L}(q, \dot{q})}{\partial q} = F$$

3) *External forces* $F = -\frac{\partial \mathcal{F}(\dot{q})}{\partial \dot{q}} + G(q)u$, due to dissipation, captured by the Rayleigh function $\mathcal{F}(\dot{q})$, and s.t.

$$\mathcal{F}(0) = 0, \ \dot{q}^\top \frac{\partial \mathcal{F}(\dot{q})}{\partial \dot{q}} > 0,$$

and interaction with environment $G(q)u$, with $u \in \mathcal{R}^m$ and $G(q) \in I\!R^{n \times m}$.

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[a] Legendre: *"Ours is the best of all worlds."*

- **EL equations**:

$$\underbrace{M\ddot{q} + \dot{M}\dot{q}}_{\dot{p}} + \underbrace{\frac{\partial}{\partial q}[\frac{1}{2}\dot{q}^\top M(q)\dot{q} + V(q)]}_{\frac{\partial H}{\partial q}} + \frac{\partial \mathcal{F}(\dot{q})}{\partial \dot{q}} = G(q)u$$

- **Hamiltonian model**: Define momenta $p = M(q)\dot{q}$ and total energy:

$$\boxed{H(q,p) = \tfrac{1}{2}p^\top M^{-1}(q)p + V(q)}$$

From

$$\dot{q} = M^{-1}(q)p = \frac{\partial H}{\partial p}$$

and the EL equations

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & -R \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ G(q) \end{bmatrix} u$$

where we assumed $\mathcal{F}(\dot{q}) = \frac{1}{2}\dot{q}^\top R\dot{q}, R = R^\top \geq 0$

- **Passivity**

$$\dot{H} = \left(\frac{\partial H}{\partial p}\right)^\top G(q)u - \left(\frac{\partial H}{\partial p}\right)^\top R\frac{\partial H}{\partial p}$$

Hence, passive outputs $\boxed{y = G^\top(q)\dot{q}}$.

- **EB controllers** (For simplicity, assume $m = n, G(q) = I$.)

We want

$$-\int_0^t \beta^\top[q(s), \dot{q}(s)]\dot{q}(s)ds = H_a[q(t), \dot{q}(t)] + \kappa$$

$$\Leftrightarrow \;\; \dot{H}_a = -\beta^\top \dot{q}.$$

Thus we can assign *any* $H_a(q)$ with

$$\beta(q) = -\frac{\partial H_a}{\partial q}(q)$$

Let $H_a(q) = -V(q) + \frac{1}{2}(q - q_*)^\top K_p(q - q_*)$, yielding

$$\beta(q) = \frac{\partial V}{\partial q}(q) - K_p(q - q_*)$$

- New total energy for $v \mapsto \dot{q}$ is

$$H_d(q, \dot{q}) = \frac{1}{2}\dot{q}^\top M(q)\dot{q} + \frac{1}{2}(q - q_*)^\top K_p(q - q_*)$$

which has a minimum in $(q_*, 0)$.

- Asymptotic stability with $v = -K_{di}\dot{q}, \; K_{di} > 0$.

B. Implications of EBE for (f, g, h) systems

Fact If the syst.

$$\Sigma : \begin{cases} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{cases}$$

satisfies the (EBE) with energy function $H(x)$. Then[a]

$$\begin{aligned} \frac{\partial H}{\partial x}^\top (x) f(x) &\leq 0 \\ g^\top (x) \frac{\partial H}{\partial x}(x) &= h(x) \end{aligned}$$

Proof Differential form of (EBE)

$$\dot{H} \leq u^\top y$$

equivalent to

$$\frac{\partial H}{\partial x}^\top (x) f(x) + \left(\frac{\partial H}{\partial x}^\top (x) g(x) - h(x) \right) u \leq 0$$

[a] Actually, *iff* as shown in *(Hill/Moylan'76)*

Corollary (EB controllers)

If

(i) Σ satisfies the (EBE).

(ii) The partial differential equation

$$\left(\frac{\partial H_a}{\partial x}(x)\right)^\top [f(x)+g(x)\beta(x)] = -\left(\frac{\partial H}{\partial x}(x)\right)^\top g(x)\beta(x)$$

can be solved for $H_a(x)$.

(iii) The total energy $H_d(x) = H(x) + H_a(x)$ has a minimum at x_*.

Then,

$$\boxed{u = \beta(x) + v \text{ is an EB–PBC}}$$

- **Proof**

$$PDE \Leftrightarrow \dot{H}_a = u^\top y.$$

$\diamond \diamond \diamond$

Caveat emptor

Limited interest because:

– (f, g, h) are cryptic models $\Rightarrow$ difficult to incorporate prior information to solve the PDE.

– Besides mechanical systems, the applicability of EB–PBC is severely stymied by the systems natural *dissipation*.

4. Dissipation obstacle for EBC

Fact A necessary condition for the solvability of the PDE is

$$f(\bar{x}) + g(\bar{x})\beta(\bar{x}) = 0 \Rightarrow h^\top(\bar{x})\beta(\bar{x}) = 0.$$

Thus, extracted power ($= h^\top \beta$) should be zero at equilibrium.

$\Rightarrow$ EB–PBC applicable only for systems with *finite dissipation*.

- OK in regulation of mechanical syst. where $power = F^\top \dot{q}$, but very restrictive for electrical or electromechanical syst.: $power = v^\top i$.

- For LTI systems (with $|A| \neq 0$)

$$\boxed{u_*^\top y_* = 0 \textit{ iff } \Sigma(0) = 0}$$

where $\Sigma(s) = C(sI - A)^{-1}B$.

A. Finite dissipation example

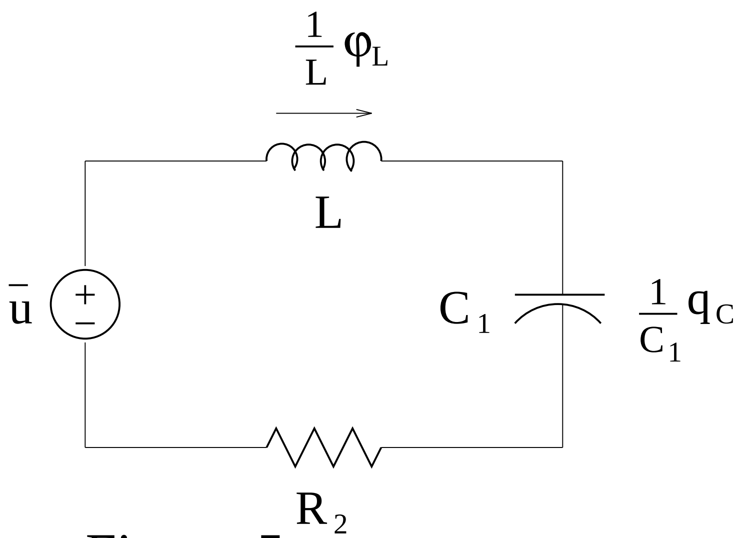

Figure 5: Series RLC circuit

State $x \stackrel{\triangle}{=} [q_C, \phi_L]^\top$, energy $H(x) = \frac{1}{2C}x_1^2 + \frac{1}{2L}x_2^2$.

- Dynamic equations

$$\Sigma : \begin{cases} \dot{x}_1 &= \frac{1}{L}x_2 \\ \dot{x}_2 &= -\frac{1}{C}x_1 - \frac{R}{L}x_2 + u \\ y &= \frac{1}{L}x_2 \end{cases}$$

- **Remarks**:

– Equil: $x_* = [x_{1*}, 0]^\top \Rightarrow$ *zero extracted power!*

– Only need to “shape” x_1

- **PDE** $\Leftrightarrow$

$$\left(\frac{1}{L}x_2\right)\frac{\partial H_a}{\partial x_1} - \left[\frac{1}{C}x_1 + \frac{R}{L}x_2 - \beta(x)\right]\frac{\partial H_a}{\partial x_2} = -\frac{1}{L}x_2\beta(x)$$

Look solution $H_a = H_a(x_1) \;\Rightarrow \beta(x_1) = -\frac{\partial H_a}{\partial x_1}(x_1)$.

- Propose

$$H_a(x_1) = \frac{1}{2C_a}x_1^2 - \left(\frac{1}{C} + \frac{1}{C_a}\right)x_{1*}x_1 + \kappa \Rightarrow$$

with C_a tuning parameter.

Recalling $H(x) = \frac{1}{2C}x_1^2 + \frac{1}{2L}x_2^2$, this yields

$$H_d(x) = \frac{1}{2}\left(\frac{1}{C} + \frac{1}{C_a}\right)(x_1 - x_{1*})^2 + \frac{1}{2L}x_2^2 + \kappa$$

has a minimum at x_* for all gains $\frac{1}{C_a} > -\frac{1}{C}$.

- **Control law**

$$u = -\frac{1}{C_a}x_1 + \left(\frac{1}{C} + \frac{1}{C_a}\right)x_{1*} \quad \left(= -\frac{1}{C_a}(x_1 - x_{1*}) + u_*\right)$$

is an $\boxed{\textbf{EB–PBC}}$.

B. Infinite dissipation example

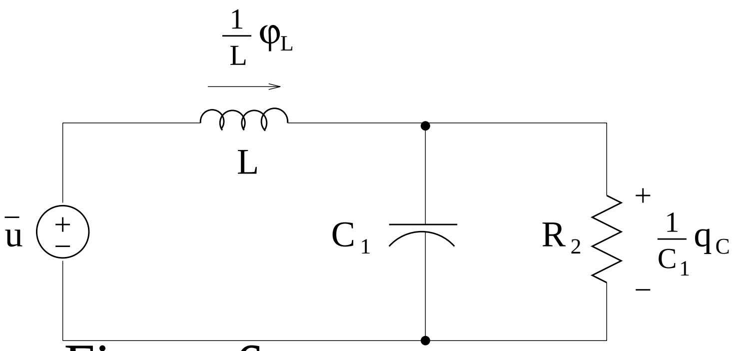

Figure 6: Parallel RLC circuit

- Model

$$\Sigma: \begin{cases} \dot{x}_1 &= -\frac{1}{RC}x_1 + \frac{1}{L}x_2 \\ \dot{x}_2 &= -\frac{1}{C}x_1 + u \\ y &= \frac{1}{L}x_2 \end{cases}$$

- **Remarks**

– Only the dissipation has changed.

– $x_* = [Cu_*, \frac{L}{R}u_*]^\top \Rightarrow$ *nonzero power* $(\forall u_* \neq 0) \Rightarrow$

$$\boxed{\lim_{t\to\infty} |\int_0^t u(s)y(s)ds| = \infty}$$

for *any* stabilizing controller (run down the battery!)

Remarks

- The dissipation obstacle is “coordinate–free”.
- In the LTI case we can design an EB–PBC on *incremental* states. Not feasible –and actually unnatural– for the general nonlinear case.
- We will propose a method (IDA–PBC) handles infinite dissipation, does not rely on incremental dynamics, and energy functions will be (in general) non–quadratic.

5. Hamiltonian models

- To characterize *admissible dissipations*: (i) Adopt a "control–as–interconnection" viewpoint, (ii) Give more structure to system

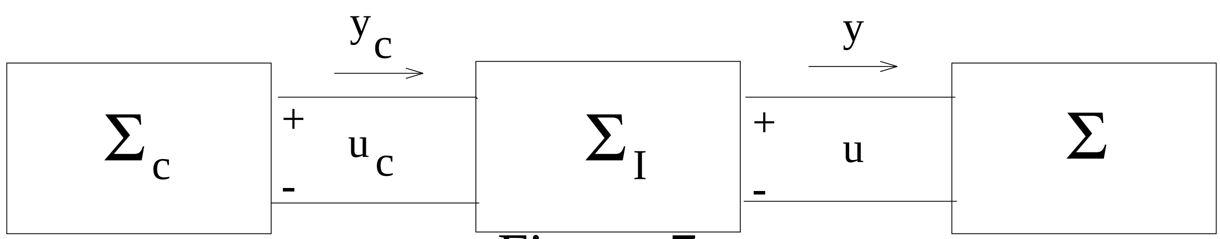

Figure 7:

Subsystems: Σ_c control, Σ_I interconnection and Σ plant.

- **Idea:** Select Σ_I such that we can *"add" the energies* of Σ and Σ_c.

- **Definition** The interconnection is *power preserving* if

$$\int_0^t [y^\top(s), y_c^\top(s)] \begin{bmatrix} u(s) \\ u_c(s) \end{bmatrix} ds = 0$$

$\diamond \diamond \diamond$

- Simplest example: Classical feedback interconnection

$$\begin{bmatrix} u \\ u_c \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ y_c \end{bmatrix},$$

A. Passive controllers

Proposition Σ_I power preserving, $\Sigma,\ \Sigma_c$ passive[a] with states $x \in I\!R^n,\ \zeta \in I\!R^{n_c}$, and energy–functions $H(x)$, $H_c(\zeta)$, resp. Let,

$$\begin{bmatrix} u \\ u_c \end{bmatrix} = \Sigma_I \begin{bmatrix} y \\ y_c \end{bmatrix} + \begin{bmatrix} v \\ v_c \end{bmatrix}$$

with (v, v_c) external inputs.

Then, $[v^\top, v_c^\top]^\top \mapsto [y^\top, y_c^\top]^\top$ is passive with *new energy–function*

$$\boxed{H(x) + H_c(\zeta)}.$$

◇ ◇ ◇

- Although $H_c(\zeta)$ is free, not clear how to affect x?

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[a]They satisfy the EBE.

B. Invariant functions method [a]

- *Idea*: restrict the motion to a subspace of (x, ζ), say

$$\Omega \triangleq \{(x, \zeta) | \zeta = F(x) + \kappa\}$$

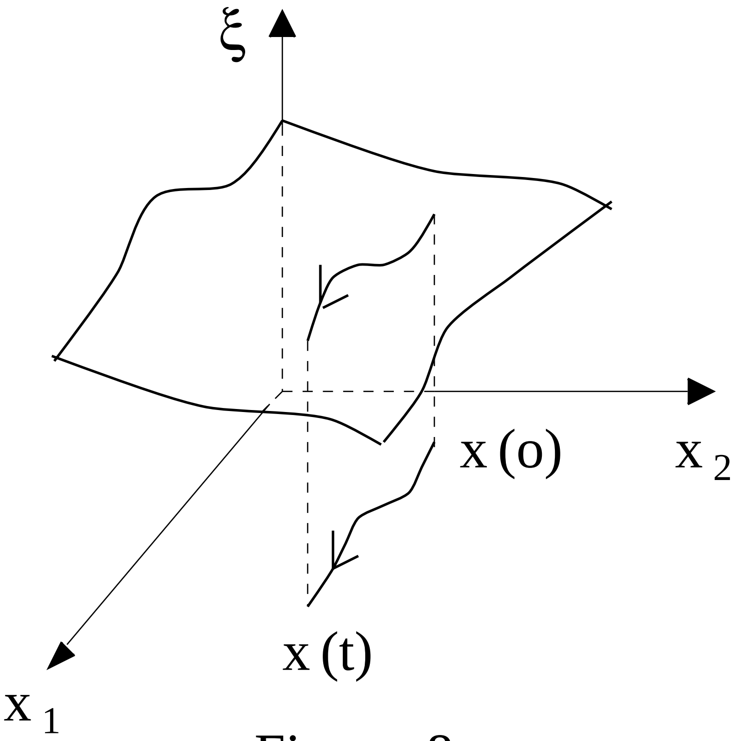

Figure 8:

Then,

$$H_d(x) \triangleq H(x) + H_c[F(x)]^{\text{b}}$$

It can be *shaped* selecting $H_c(\zeta)$.

- Find $F(\cdot)$ that renders Ω *invariant*

[a] *(Marsden/Ratiu,'94; Dalsmo/van der Schaft,'99)*

[b] κ determined by the controllers ICs, w.l.o.g. $\kappa = 0$.

Series RLC circuit

- Controller an integrator

$$\Sigma_c : \begin{cases} \dot{\zeta} & = & u_c \\ y_c & = & \frac{\partial H_c}{\partial \zeta}(\zeta) \end{cases}$$

with negative feedback interconnection
$u = -y_c, u_c = y$.

- Invariant function candidate: $\mathcal{C}(x_1, \zeta) \stackrel{\triangle}{=} F(x_1) - \zeta$.

Look for $F(\cdot)$ s.t. $\frac{d}{dt}\mathcal{C} \equiv 0$. Recalling

$$\Sigma : \begin{cases} \dot{x}_1 & = & \frac{1}{L}x_2 \\ \dot{x}_2 & = & -\frac{1}{C}x_1 - \frac{R}{L}x_2 + u \\ y & = & \frac{1}{L}x_2 \end{cases}$$

Now,

$$\frac{d}{dt}\mathcal{C} = \frac{1}{L}x_2 \left(\frac{\partial F}{\partial x_1}(x_1) - 1 \right)$$

Thus, we take $F(x_1) = x_1$.

With the controller energy

$$H_c(\zeta) = \frac{1}{2C_a}\zeta^2 - \left(\frac{1}{C} + \frac{1}{C_a} \right) x_{1*}\zeta$$

we recover the previous $H_d(x)$.

Remarks

- EB–PBC is constant voltage source in series with a capacitor C_a. The control action can be implemented *without* the addition of dynamics.

- We have assumed that $n_c = n$. We also have considered $n_c \neq n$ and other Σ_I.

- Stabilization is ensured for all $C_a > -C$, but the system Σ_c is passive only for positive values of C_a.

- Finding $F(\cdot)$ that renders Ω invariant involves the *solution of a PDE*. The search for a solution of the PDE can be made systematic by incorporation of additional structure—starting with the choice of a suitable system representation.

C.1 Port–controlled Hamiltonian (PCH) syst.

• To formalize geometrically power preserving interconnections we need the notion of a **Dirac structure**.

• Consider the linear space $\mathcal{F}$ of *flows* f, and its dual, $\mathcal{F}^*$, the space of *efforts* e. *Power* is $P = \langle e \mid f \rangle$, with $\langle \cdot | \cdot \rangle$ the duality product.

On $\mathcal{F} \times \mathcal{F}^*$ define

$$\ll (f_1, e_1), (f_2, e_2) \gg \stackrel{\triangle}{=} \langle e_1 \mid f_2 \rangle + \langle e_2 \mid f_1 \rangle$$

Definition Dirac structure:

$$\boxed{\mathcal{D} \stackrel{\triangle}{=} \left\{ (f, e) \in \mathcal{F} \times \mathcal{F}^* \mid \mathcal{D}^\perp = \mathcal{D} \right\}}$$

where $\mathcal{D}^\perp$ denotes the orthogonal complement (with respect to the bilinear form.)

Corollary

For all $(f, e) \in \mathcal{D} = \mathcal{D}^\perp$, we have that

$$0 = \ll (f, e), (f, e) \gg = 2\langle f \mid e \rangle,$$

therefore $P = 0$ for all elements of $\mathcal{D}$. Hence, a Dirac structure defines a power conserving relation.

- The space of flows is partitioned as

$$\mathcal{F} \stackrel{\triangle}{=} \mathcal{F}_S \times \mathcal{F}_R \times \mathcal{F}_P,$$

with $\mathcal{F}_S, \mathcal{F}_R, \mathcal{F}_P$– space of flows f_S, f_R, f_P connected to the energy storing, dissipative and external elements (connected to the environment). Analogously define $\mathcal{F}^* \stackrel{\triangle}{=} \mathcal{F}^*_S \times \mathcal{F}^*_R \times \mathcal{F}^*_P$, for efforts e_S, e_R, e_P. Power then becomes

$$P = \langle e_S \mid f_S \rangle + \langle e_R \mid f_R \rangle + \langle e_P \mid f_P \rangle = 0$$

- In *PCH systems* the Dirac structure is

$$\begin{aligned} f_S &= -J(x)e_S - g_R(x)f_R - g(x)f_P \\ e_P &= g^\top(x)e_S \\ e_R &= g_R^\top(x)e_S \end{aligned}$$

where $J(x) = -J^\top(x)$ is the interconnection matrix, and $g(x),\ g_S(x),\ g_R(x)$ are input matrices. Clearly, $\mathcal{D} = \mathcal{D}^\perp$.

- *If* $f_R = -R(x)e_R$, where $R(x) = R^\top(x) \geq 0$,

$$\begin{bmatrix} f_S \\ e_P \end{bmatrix} = \begin{bmatrix} -J(x) + \mathcal{R}(x) & -g(x) \\ g^\top(x) & 0 \end{bmatrix} \begin{bmatrix} e_S \\ f_P \end{bmatrix}$$

where $\mathcal{R}(x) := g_R(x)R(x)g_R^\top(x) \geq 0$.

Now, energy $H(x)$, is such that

$$P(t) = \frac{d}{dt} H[x(t)] = \langle \frac{\partial H}{\partial x}[x(t)] \mid \dot{x}(t) \rangle \quad \equiv \langle e_s \mid f_s \rangle$$

hence

$$\begin{array}{rcl} f_S & = & -\dot{x} \\ e_S & = & \frac{\partial H}{\partial x}(x) \end{array}$$

We get the well–known model of a PCH system

$$\Sigma \; : \; \left\{ \begin{array}{rcl} \dot{x} & = & [J(x) - \mathcal{R}(x)] \frac{\partial H}{\partial x}(x) + g(x)u \\ y & = & g^{\top}(x) \frac{\partial H}{\partial x}(x) \end{array} \right.$$

where $u \triangleq f_P, y \triangleq e_P$.

- PCH systems clearly satisfy the EBE

$$\frac{d}{dt} H[x(t)] = -\frac{\partial^{\top} H}{\partial x}[x(t)] \mathcal{R}[x(t)] \frac{\partial H}{\partial x}[x(t)] + u^{\top}(t) y(t)$$

C.2 Examples

- *Series RLC Circuit*

– State: $x = [q_C, \phi_L]^\top$, total energy:

$$H(x) = \frac{1}{2}\frac{x_1^2}{C} + \frac{1}{2}\frac{x_2^2}{L}$$

– Power variables representation: We have

$$f_S = -\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}, e_S = \begin{bmatrix} \frac{x_1}{C} \\ \frac{x_2}{L} \end{bmatrix}$$

$$g_R(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathcal{R}(x) = \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix}, f_R = -Re_R$$

$$\begin{bmatrix} f_S \\ y \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & R \end{bmatrix} & -\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \end{bmatrix} & 0 \end{bmatrix} \begin{bmatrix} e_S \\ u \end{bmatrix}$$

– PCH model

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -R \end{bmatrix}}_{J-\mathcal{R}} \underbrace{\begin{bmatrix} \frac{x_1}{C} \\ \frac{x_2}{L} \end{bmatrix}}_{\frac{\partial H}{\partial x}} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{g} u$$

- **Mechanical systems**

– Assuming linear *friction*,

$$F = R\dot{q}$$

where, $R = R^\top \geq 0$

– State $x = \begin{bmatrix} q \\ p \end{bmatrix}$, $p \triangleq D(q)\dot{q}$ momenta.

– Total energy:

$$H(q,p) = \frac{1}{2}p^\top D^{-1}(q)p + U(q)$$

– PCH model,

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & I \\ -I & -R \end{bmatrix} \frac{\partial H}{\partial x}(x) + \begin{bmatrix} 0 \\ I \end{bmatrix} u \\ y &= \begin{bmatrix} 0 \\ I \end{bmatrix} \frac{\partial H}{\partial x}(x) \left(= D^{-1}(q)p \right) \end{aligned}$$

- *Electromechanical systems*

– Assuming linear magnetics, i.e., $\lambda = L(\theta)i \in I\!R^n$, $L(\theta) = L^\top(\theta) \geq 0$, one mechanical d.o.f. $\theta \in I\!R$, $u \in I\!R^m$ voltages.

– Total energy:

$$H = \frac{1}{2}\lambda^\top L^{-1}(\theta)\lambda + \frac{m}{2}\dot{\theta}^2 + U(\theta)$$

– State $x = \begin{bmatrix} \lambda \\ \theta \\ m\dot{\theta} \end{bmatrix}$, $\frac{\partial H}{\partial x}(x) = \begin{bmatrix} i \\ -\tau \\ \dot{\theta} \end{bmatrix}$, τ force

– PCH model

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -R & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \frac{\partial H}{\partial x}(x) + \begin{bmatrix} Mu \\ 0 \\ -\tau_L \end{bmatrix} \\ y &= M\frac{\partial H}{\partial x_1}(x)(= Mi) \end{aligned}$$

where $\tau_L \in I\!R$ load torque, $M \in I\!R^{n\times m}$ defines actuated coordinates.

Induction motor

We have $n = 4,\ m = 2$,

$$\lambda = \begin{bmatrix} \lambda_s I \\ \lambda_r I \end{bmatrix},\ M = \begin{bmatrix} I \\ 0 \end{bmatrix},$$

$$L(\theta) = \begin{bmatrix} L_s & L_{sr}e^{J\theta} \\ L_{sr}e^{J\theta} & L_s \end{bmatrix}, R = \begin{bmatrix} R_s I & 0 \\ 0 & R_r I \end{bmatrix}$$

- *Power converters* [a]

$\Rightarrow$ More general class of PCH models:

$$\dot{x} = [J(x,u) - \mathcal{R}(x)]\frac{\partial H}{\partial x} + g(x,u)$$

The control u modifies the *interconnection*

– Assuming: fast switching, linear R_i, L_i, C_i.

– State $x \triangleq \begin{bmatrix} \phi_L \\ q_C \end{bmatrix}$

– Total energy: $H(x) = \frac{1}{2}x_1^\top \mathbf{L}^{-1} x_1 + \frac{1}{2}x_2^\top \mathbf{C}^{-1} x_2$,
where $\mathbf{L} = \mathit{diag}\{L_i\}, \mathbf{C} = \mathit{diag}\{C_i\}$.

– **Boost** PCH model ($x \in I\!\!R^2$):

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & -u \\ u & \frac{-1}{R} \end{bmatrix}}_{J(u)-\mathcal{R}} \frac{\partial H}{\partial x}(x) + \begin{bmatrix} E \\ 0 \end{bmatrix}$$

[a](Escobar et al., Automatica '99)

– **Cuk** PCH model ($x \in I\!R^4$):

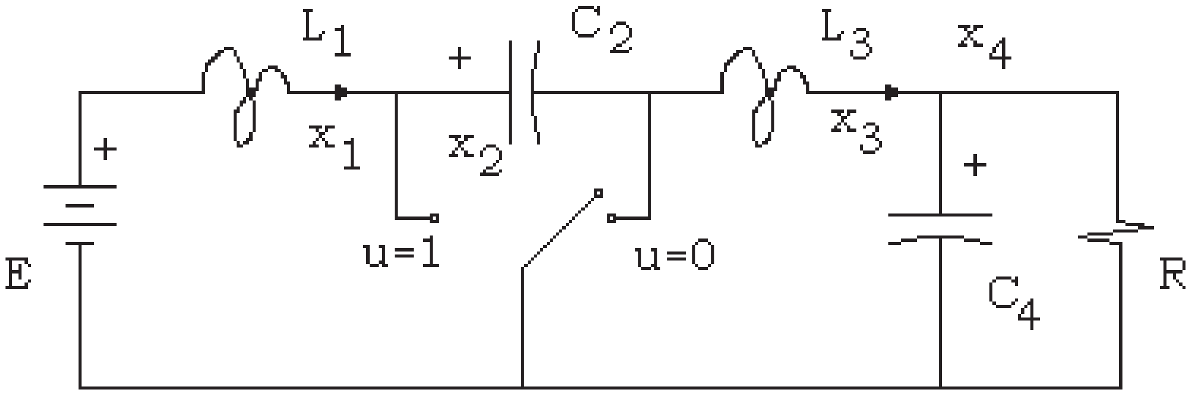

Figure 9:

$$\dot{x} = \begin{bmatrix} 0 & -(1-u) & 0 & 0 \\ 1-u & 0 & u & 0 \\ 0 & -u & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{R} \end{bmatrix} \frac{\partial H}{\partial x}(x) + \begin{bmatrix} E \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

C.3 Energy balancing PBC of PCH systems

Consider PCH controllers

$$\Sigma_c \ : \ \left\{ \begin{array}{rcl} \dot{\zeta} & = & [J_c(\zeta) - \mathcal{R}_c(\zeta)]\frac{\partial H_c}{\partial \zeta}(\zeta) + g_c(\zeta)u_c \\ y_c & = & g_c^\top(\zeta)\frac{\partial H_c}{\partial \zeta}(\zeta) \end{array} \right.$$

$J_c(\zeta) = -J_c^\top(\zeta)$, $\mathcal{R}_c(\zeta) = \mathcal{R}_c^\top(\zeta) \geq 0$, and standard feedback interconnection, i.e., $u = -y_c, u_c = y$.

- Closed–loop is still PCH:

$$\left[\begin{array}{c} \dot{x} \\ \dot{\zeta} \end{array} \right] = \left[\begin{array}{cc} J(x) - \mathcal{R}(x) & -g(x)g_c^\top(\zeta) \\ g_c(\zeta)g^\top(x) & J_c(\zeta) - \mathcal{R}_c(\zeta) \end{array} \right] \left[\begin{array}{c} \frac{\partial H}{\partial x}(x) \\ \frac{\partial H_c}{\partial \zeta}(\zeta) \end{array} \right]$$

with total energy $H(x) + H_c(\zeta)$.

- **Casimir functions**: conserved quantities of the system *for any* choice of the Hamiltonian; completely determined by the geometry (i.e., the interconnection structure) of the system.

Look for $\mathcal{C}(x,\zeta) = F(x) - \zeta$, such that $\frac{d}{dt}\mathcal{C} = 0$ *for all* $H(x)$, hence

$$\left[\begin{array}{cc} \frac{\partial F^\top}{\partial x} & -I_m \end{array}\right] \left[\begin{array}{cc} J(x) - \mathcal{R}(x) & -g(x)g_c^\top(\zeta) \\ g_c(\zeta)g^\top(x) & J_c(\zeta) - \mathcal{R}_c(\zeta) \end{array}\right] = 0.$$

Proposition $\mathcal{C}(x,\zeta)$ is a Casimir function *if and only if* $F(x)$ satisfies

$$\begin{aligned} \left(\frac{\partial F}{\partial x}(x)\right)^\top J(x) \frac{\partial F}{\partial x}(x) &= J_c(\zeta) \\ \mathcal{R}(x)\frac{\partial F}{\partial x}(x) &= 0 \qquad (*) \\ \mathcal{R}_c(\zeta) &= 0 \\ \left(\frac{\partial F}{\partial x}(x)\right)^\top J(x) &= g_c(\zeta)g^\top(x) \end{aligned}$$

Dynamics reduced to Ω :

$$\Sigma_d : \boxed{\dot{x} = [J(x) - \mathcal{R}(x)]\frac{\partial H_d}{\partial x}(x)}$$

with $H_d(x) = H(x) + H_c[F(x) + \kappa]$. $\diamond\diamond\diamond$

D. Admissible dissipation

(*) implies that, for any $H_c(\cdot)$,

$$\mathcal{R}(x)\frac{\partial H_c(F)}{\partial x}(x) = 0$$

Assume $\mathcal{R}(x)$ diagonal,[a] then H_c should not depend on coordinates where there is damping. Consequently:

Dissipation only in "non–shaped" coordinates.

Remark

In mechanical systems, damping appears in velocities, and we only "shape" positions. In series RLC resistance is in "non–shaped" x_2; while in the parallel RLC both coordinates have to be shaped.

[a] Must often encountered in applications.

6. IDA PB Control

- How to overcome dissipation obstacle?

♡ *Explicitly* incorporating information on the state, obviates the need of the Casimir functions and still shape the energy function.

A. Matching perspective

- New total energy

$$H_d(x) = H(x) + H_a(x)$$

and desired dynamics

$$\dot{x} = [J(x) - \mathcal{R}(x)]\frac{\partial H_d}{\partial x}$$

- Find $\beta(x)$ such that the PDE is solved (for $H_a(x)$)

(PDE) $\boxed{[J(x) - \mathcal{R}(x)]\frac{\partial H_a}{\partial x}(x) = g(x)\beta(x)}$

and setting $u = \beta(x)$, yields

$$\begin{aligned} \dot{x} &= [J(x) - \mathcal{R}(x)]\frac{\partial H}{\partial x}(x) + g(x)\beta(x) \\ &= [J(x) - \mathcal{R}(x)]\underbrace{\left(\frac{\partial H}{\partial x}(x) + \frac{\partial H_a}{\partial x}(x)\right)}_{\frac{\partial H_d}{\partial x}} \end{aligned}$$

If, further, $\boxed{x_* = \arg\min H_d(x)}$ then x_* is stable.

B. Control as a state–modulated source

- For *infinite dissipation* syst. need non–passive controllers!

1) Controller as an (infinite energy) *source*

$$\Sigma_c : \left\{ \begin{array}{rcl} \dot{\zeta} & = & u_c \\ y_c & = & \frac{\partial H_c}{\partial \zeta}(\zeta) \end{array} \right.$$

with energy function $H_c(\zeta) = -\zeta$.

2) *State–modulated* (power–preserv.) interconnection

$$\left[\begin{array}{c} u(s) \\ u_c(s) \end{array} \right] = \left[\begin{array}{cc} 0 & -\beta(x) \\ \beta(x) & 0 \end{array} \right] \left[\begin{array}{c} y(s) \\ y_c(s) \end{array} \right]$$

– Overall interconnected PCH system (with total energy $H(x) + H_c(\zeta)$)

$$\left[\begin{array}{c} \dot{x} \\ \dot{\zeta} \end{array} \right] = \left[\begin{array}{cc} J(x) - \mathcal{R}(x) & -g(x)\beta(x) \\ \beta^\top(x) g^\top(x) & 0 \end{array} \right] \left[\begin{array}{c} \frac{\partial H}{\partial x}(x) \\ \frac{\partial H_c}{\partial \zeta}(\zeta) \end{array} \right]$$

C. Parallel RLC circuit

- Total energy: $H(x) = \frac{1}{2}\frac{x_1^2}{C} + \frac{1}{2}\frac{x_2^2}{L}$
- PCH model

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ \mathcal{R} = \begin{bmatrix} 1/R & 0 \\ 0 & 0 \end{bmatrix},\ g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(**) becomes

$$\begin{aligned} -\frac{1}{R}\frac{\partial H_a}{\partial x_1}(x) + \frac{\partial H_a}{\partial x_2}(x) &= 0 \\ -\frac{\partial H_a}{\partial x_1}(x) &= \beta(x) \end{aligned}$$

First equation is a PDE with solution

$$H_a(x) = \Phi(Rx_1 + x_2)$$

where $\Phi(\cdot)$ arbitrary.

- Choose $\Phi(\cdot)$ so that

$$\underbrace{\begin{bmatrix} C \\ \frac{L}{R} \end{bmatrix} u_*}_{x_*} = \arg\min H_d(x) \Leftarrow \begin{cases} \frac{\partial H_d}{\partial x}(x_*) = 0 \quad (EC) \\ \frac{\partial^2 H_d}{\partial x^2}(x_*) > 0 \quad (HC) \end{cases}$$

- (EC) equivalent

$$\frac{\partial H_a}{\partial x}(x_*) = \begin{bmatrix} R \\ 1 \end{bmatrix} \frac{\partial \Phi}{\partial z}(z_*) \equiv -\frac{\partial H}{\partial x}(x_*) = \begin{bmatrix} 1 \\ \frac{1}{R} \end{bmatrix} u_*$$

where $z = Rx_1 + x_2$. Thus, $\frac{\partial \Phi}{\partial z}(z_*) = -\frac{1}{R}u_*$.

- Check (HC)

$$\frac{\partial^2 H_a}{\partial x^2} = \frac{\partial^2 \Phi}{\partial z^2} \begin{bmatrix} R^2 & R \\ R & 1 \end{bmatrix}$$

- Let

$$\Phi(z) = \frac{K_p}{2}(z - z_*)^2 - \frac{1}{R}u_* z$$

which yields

$$H_d(x) = (x - x_*)^\top \begin{bmatrix} \frac{1}{C} + R^2 K_p & RK_p \\ RK_p & \frac{1}{L} + K_p \end{bmatrix} (x - x_*) + \kappa$$

(HC) satisfied for $K_p > \frac{-1}{(L + CR^2)}$

- Control

$$u = -K_p[R(x_1 - x_{1*}) + x_2 - x_{2*}] + u_*.$$

Remark

From **(PDE)** *results* a family of "admissible" $H_a(x), \beta(x)$. Choose one that shapes the energy.

D. Interconnection and damping assignment

- We aim at

$$\dot{x} = [J_d(x) - \mathcal{R}_d(x)] \frac{\partial H_d}{\partial x}(x)$$

for some *new* $J_d(x) = -J_d^\top(x),\ \mathcal{R}_d(x) = \mathcal{R}_d^\top(x) \geq 0$.

- **(PDE)** becomes

$$[J(x) + J_a(x) - \mathcal{R}(x) - \mathcal{R}_a(x)] \frac{\partial H_a}{\partial x} =$$

$$-[J_a(x) - \mathcal{R}_a(x)] \frac{\partial H}{\partial x} + g(x)\beta(x) \quad (PDE')$$

where $J_a(x) \triangleq J_d(x) - J(x),\ \ \mathcal{R}_a(x) \triangleq \mathcal{R}_d(x) - \mathcal{R}(x)$ are new *degrees of freedom*.

E. Solving the PDE

- **(PDE')**

$\Leftrightarrow \boxed{g^{\perp}[J_d - \mathcal{R}_d]\frac{\partial H_a}{\partial x} = -g^{\perp}[J_a - \mathcal{R}_a]\frac{\partial H}{\partial x}}$,

where $g^{\perp}(x)g(x) = 0$.

Control:

$$\beta(x) = [g^{\top}g]^{-1}g^{\top}\left\{[J_d - \mathcal{R}_d]\frac{\partial H_a}{\partial x} + [J_a - \mathcal{R}_a]\frac{\partial H}{\partial x}\right\}$$

- **Lemma** Given $K(x) : I\!\!R^n \to I\!\!R^n$.

$$\exists H_a(x) : I\!\!R^n \to I\!\!R | K(x) = \frac{\partial H_a}{\partial x} \Leftrightarrow \frac{\partial K}{\partial x} = \left[\frac{\partial K}{\partial x}\right]^{\top} \; (IC)$$

Consequently, we can equivalently check that

$$K(x) \stackrel{\triangle}{=} -[J_d - \mathcal{R}_d]^{-1}\left([J_a - \mathcal{R}_a]\,\frac{\partial H}{\partial x} - g\beta\right)$$

satisfies (IC), which defines a PDE *directly* for β.

F. When is IDA an EB–PBC?

Compute

$$\dot{H}_d = \underbrace{u^\top y - \left[\frac{\partial H}{\partial x}(x)\right]^\top \mathcal{R}(x)\frac{\partial H}{\partial x}(x)}_{\dot{H}} + \dot{H}_a$$

$$= -\left[\frac{\partial H_d}{\partial x}(x)\right]^\top \mathcal{R}_d(x)\frac{\partial H_d}{\partial x}(x)$$

and $\mathcal{R}_d(x) = \mathcal{R}_a(x) + \mathcal{R}(x)$, we have that

$$\dot{H}_a = -u^\top y - \left[2\frac{\partial H}{\partial x} + \frac{\partial H_a}{\partial x}\right]^\top I\!R \frac{\partial H_a}{\partial x} - \left[\frac{\partial H_d}{\partial x}\right]^\top \mathcal{R}_a \frac{\partial H_d}{\partial x}.$$

Consequently, if $\boxed{\mathcal{R}_a(x) = 0}$ and the natural damping $\mathcal{R}(x)$ satisfies the condition

$$\mathcal{R}(x)\frac{\partial H_a}{\partial x}(x) = 0,$$

then

$$\boxed{\dot{H}_a = -u^\top y}$$

$\Leftrightarrow$ IDA–PBC is *energy balancing.*

G. Universal stabilizing property of IDA–PBC

Proposition

If $\exists \beta(x) \in \mathcal{C}^1$ that asymptotically stabilizes the PCH system, then $\exists J_a(x), \mathcal{R}_a(x) \in \mathcal{C}^0$ and $H_a(x) \in \mathcal{C}^1$ which satisfy the conditions of the IDA–PBC theorem.

$\Leftrightarrow$ IDA–PBC methodology generates *all* asymptotically stabilizing controllers for PCH systems.

Lemma

If x_* is *asymptotically stable* for $\dot{x} = f(x),\ f(x) \in \mathcal{C}^1$ then $\exists H_d(x) \in \mathcal{C}^1$, positive definite, and $\mathcal{C}^0$ functions

$$J_d(x) = -J_d^\top(x), \mathcal{R}_d(x) = \mathcal{R}_d^\top(x) \geq 0$$

such that

$$f(x) = [J_d(x) - \mathcal{R}_d(x)] \frac{\partial H_d}{\partial x}$$

Proof Converse Lyapunov theorem $\Rightarrow \exists H_d(x)$ s.t.

$$\left[\frac{\partial H_d}{\partial x}(x)\right]^\top f(x) \leq 0.$$

Define

$$\begin{aligned} \mathcal{R}_d(x) &:= -\frac{1}{|\frac{\partial H_d}{\partial x}|^4} \frac{\partial H_d}{\partial x} \left[\frac{\partial H_d}{\partial x}\right]^\top f^\top(x) \frac{\partial H_d}{\partial x} \\ J_d(x) &:= \frac{1}{|\frac{\partial H_d}{\partial x}|^2} \left\{ f(x) \left[\frac{\partial H_d}{\partial x}\right]^\top - \frac{\partial H_d}{\partial x} f^\top(x) \right\} \end{aligned}$$

H. Integral action

- Adding an integrator around passive output

Proposition

IDA–PBC with integral action

$$u = u_{es} + u_{di} + v$$

where

$$\dot{v} \quad = \quad -K_I g^\top \frac{\partial H_d}{\partial x}$$

with $K_I = K_I^\top > 0$, preserves stability.

Proof

Let

$$W(x, u_{di}) \triangleq H_d + \frac{1}{2} v^\top K_I^{-1} v.$$

The closed–loop

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} J_d - \mathcal{R}_d & gK_I \\ -K_I g^\top & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial W}{\partial x} \\ \frac{\partial W}{\partial v} \end{bmatrix}$$

I. Output feedback with ”dirty derivatives”

- Obviate velocity measurement feeding back the dirty derivative of passive output

$$u_{di} = -\frac{k_v p}{\tau p + 1}\left(g^\top \frac{\partial H_d}{\partial x}\right)$$

Proposition

If

$$u = u_{es} + k_v g^\top \frac{\partial H_d}{\partial x}$$

is stable then the dynamic output feedback IDA–PBC $u = u_{es} + u_{di}$, where

$$\begin{aligned}\dot{z} &= -\frac{1}{\tau}z + \frac{k_v}{\tau^2} g^\top \frac{\partial H_d}{\partial x} \\ u_{di} &= z - \frac{k_v}{\tau} g^\top \frac{\partial H_d}{\partial x}\end{aligned}$$

is also stable.

Proof

$$\dot{u}_{di} = -\frac{1}{\tau}u_{di} - \frac{k_v}{\tau} g^\top \frac{\partial H_d}{\partial x}$$

Consequently, with $W(x, u_{di}) \stackrel{\triangle}{=} H_d + \frac{k_v}{2\tau}u_{di}^2$,

$$\begin{bmatrix} \dot{x} \\ \dot{u}_{di} \end{bmatrix} = \begin{bmatrix} J_d & \frac{k_v}{\tau} g \\ -\frac{k_v}{\tau} g^\top & -\frac{k_v}{\tau^2} \end{bmatrix} \begin{bmatrix} \frac{\partial W}{\partial x} \\ \frac{\partial W}{\partial u_{di}} \end{bmatrix}$$

with $\dot{W} = -\frac{1}{k_v}(u_{di})^2$.

Applications

– Mass–balance systems (ACC, 2000),

– electrical motors (IEEE CST, 2001),

– power systems (ACC, 2001),

– magnetic levitation systems (MTNS, 2000),

– underactuated mechanical systems (IEEE TAC, 2001),

– power converters (SCL, 99),

– rigid body dynamics (AIAA, 2000),

– underwater vehicles (ACC, 2001).

7.1 Magnetic levitation system

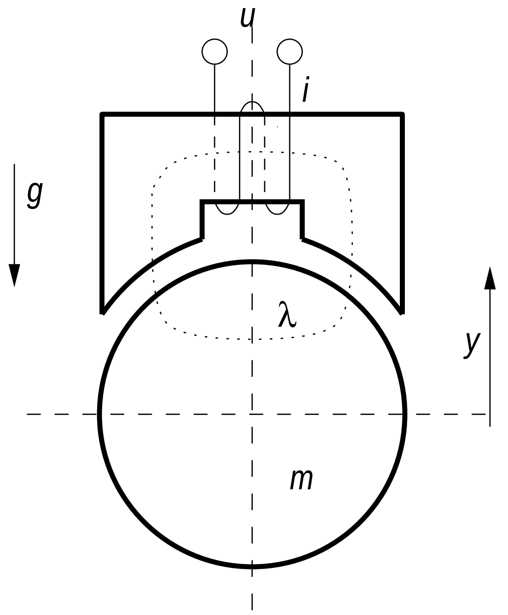

Figure 10: Levitated ball

- Approximate the inductance $L(\theta) = \frac{k}{1-\theta}$.
- **PCH model**. State: $x = [\lambda, \theta, m\dot{\theta}]^\top$; Hamiltonian:

$$H(x) = \frac{1}{2k}(1 - x_2)x_1^2 + \frac{1}{2m}{x_3}^2 + mgx_2$$

Then

$$J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \ \mathcal{R} = \begin{bmatrix} R & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ g = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Equilibrium $x_* = [\sqrt{2kmg}, x_{2*}, 0]^\top$.

Structural limitation

(PDE) without changing J or $\mathcal{R}$:

$$(J - \mathcal{R})\frac{\partial H_a}{\partial x}(x) = g\beta(x) \Leftrightarrow \left\{ \begin{array}{rcl} -R\frac{\partial H_a}{\partial x_1}(x) & = & \beta(x) \\ \frac{\partial H_a}{\partial x_2}(x) & = & 0 \\ \frac{\partial H_a}{\partial x_3}(x) & = & 0 \end{array} \right.$$

$\Rightarrow H_a = H_a(x_1)$ can only depend on x_1. The Hessian

$$\frac{\partial^2 H_d}{\partial x^2}(x) = \left[\begin{array}{ccc} \frac{(1-x_2)}{k} + \frac{\partial^2 H_a}{\partial x^2}(x_1) & -\frac{x_1}{k} & 0 \\ -\frac{x_1}{k} & 0 & 0 \\ 0 & 0 & \frac{1}{m} \end{array} \right]$$

which is *sign indefinite* for all $H_a(x_1)$.

- Source of the problem: lack of effective coupling between the electrical and the mechanical subsyst.

IDA–PBC

- Enforce a *coupling* between the flux x_1 and the velocity x_3 $\Rightarrow$

$$J_d = \begin{bmatrix} 0 & 0 & -\alpha \\ 0 & 0 & 1 \\ \alpha & -1 & 0 \end{bmatrix}$$

- **(PDE')**

$$\begin{aligned} \frac{\partial H_a}{\partial x_3}(x) &= 0 \\ -R\frac{\partial H_a}{\partial x_1}(x)(x) &= \frac{\alpha}{m}x_3 + \beta(x) \\ \alpha\frac{\partial H_a}{\partial x_1}(x) - \frac{\partial H_a}{\partial x_2}(x) &= -\frac{\alpha}{k}(1 - x_2)x_1 \end{aligned}$$

Solving the latter (e.g. **Maple**)

$$H_a(x) = \frac{1}{6k\alpha}x_1^3 + \frac{1}{2k}x_1^2(x_2 - 1) + \Phi(x_2 + \frac{1}{\alpha}x_1),$$

- Suitable choice for $\Phi(\cdot)$

$$\Phi(x_2 + \frac{1}{\alpha}x_1) = mg[-(\tilde{x}_2 + \frac{1}{\alpha}\tilde{x}_1) + \frac{b}{2}(\tilde{x}_2 + \frac{1}{\alpha}\tilde{x}_1)^2]$$

- Control law

$$u = \underbrace{\frac{R}{k}(1 - x_2)x_1}_{Ri} - \underbrace{K_p(\frac{1}{\alpha}\tilde{x}_1 + \tilde{x}_2) - \frac{\alpha}{m}x_3}_{PD} - \underbrace{\frac{R}{\alpha}(\frac{1}{2k}x_1^2 - mg)}_{undesirable}$$

- To remove the high order term: shuffle the damping

$$\mathcal{R}_a = \begin{bmatrix} -R & 0 & 0 \\ 0 & R_a & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This yields

$$\boxed{u = \tfrac{R}{k}(1 - x_2)x_1 - K_p(\tfrac{1}{\alpha}\tilde{x}_1 + \tilde{x}_2) - \left(\tfrac{\alpha}{m} + K_p\right)x_3}$$

Other control approaches

Nested–loop control

$$\begin{aligned}\dot{\lambda} + \frac{R}{k}(1-\theta)\lambda &= u\\ m\ddot{\theta} &= F - mg\\ F &= \frac{1}{2}\lambda^2\end{aligned}$$

- Can be decomposed as a feedback interconnection of electrical ($\Sigma_1 : \begin{bmatrix} u \\ \dot{\theta} \end{bmatrix} \mapsto \begin{bmatrix} \lambda \\ F \end{bmatrix}$) and mechanical ($\Sigma_2 : F - mg \mapsto \dot{\theta}$) **passive** subsystems.

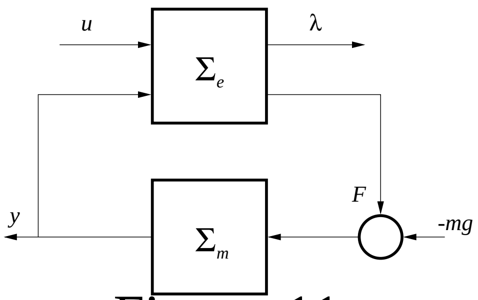

Figure 11:

Suggesting a nested–loop control configuration.

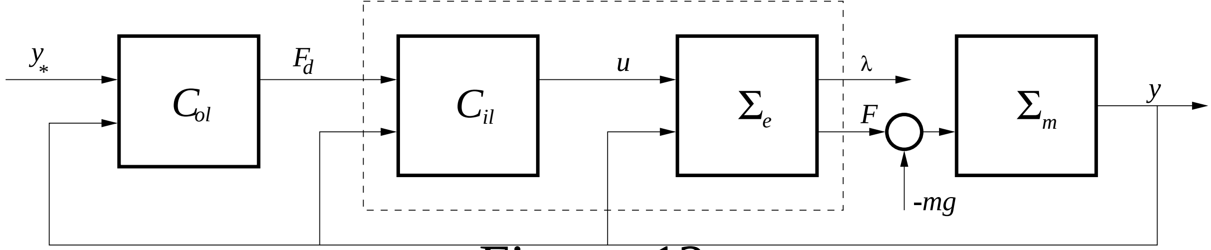

Figure 12:

1. Feedback–linearization

Introduce a change of coordinates: $F = \frac{1}{2}\lambda^2$,

$$\begin{aligned} \dot{F} &= -\tfrac{2R}{k}(1-\theta)F + \sqrt{\tfrac{2F}{k}}u \\ m\ddot{\theta} &= F - mg \end{aligned}$$

Differentiating once more $\ddot{\theta}$ we get

$$m\theta^{(3)} = \dot{F} = -R(1-\theta)\frac{2F}{k} + \sqrt{\frac{2F}{k}}u$$

With output θ, and provided $F > 0$, the system has relative degree 3, i.e. its **flat** *(Levine, et al'96)*.

- The **FLC** $u = u_{\rm FL}$ with

$$u_{\rm FL} = \sqrt{\frac{k}{2F}} m v_{\rm FL}(\theta, \dot{\theta}, F) + R(1-\theta)\sqrt{\frac{2F}{k}}$$

yields $\boxed{\theta^{(3)} = v_{\rm FL}(\theta, \dot{\theta}, F)}$.

- **Outer loop**

$$v_{\rm FL}(\theta, \dot{\theta}, F) = \theta_*^{(3)} - k_2[\underbrace{(\frac{1}{m}F - g)}_{\ddot{\theta}} - \ddot{\theta}_*] - k_1\dot{\tilde{\theta}} - k_0\tilde{\theta}$$

where $\tilde{\theta} \stackrel{\triangle}{=} \theta - \theta_*$. Overall dynamics

$$D(p)\tilde{\theta} = (p^3 + k_2 p^2 + + k_1 p + k_0)\tilde{\theta}$$

2. Standard passivity–based control

- *Steps*:

1) Assign desired storage and dissipation functions to inner–loop s.t. $\lambda \to \lambda_d$.

2) Design outer–loop s.t. $\lambda \to \lambda_d \Rightarrow F \to F_d$.

3) Prove internal stability, i.e., $F \to F_d \Rightarrow \theta \to \theta_*$.

(i) **Flux tracking**

♡ $u \mapsto \lambda$ is **output strictly passive** with storage function $H = \frac{1}{2}\lambda^2 \quad \Rightarrow \lambda$ is "easy" to control. But, rely on natural dissipation

$$\int_0^t u(s)\lambda(s)ds \geq \alpha \int_0^t \lambda^2(s)ds + \beta$$

with $\alpha \triangleq \frac{R\epsilon}{k} > 0$, where $\epsilon > 0$ is determined by the domain of validity of the model

$$-\infty < \theta \leq 1 - \epsilon.$$

Thus, **slow convergence**.

- **Desired storage function**

$$\boxed{H_d = \tfrac{1}{2}\tilde{\lambda}^2},$$

where $\tilde{\lambda} \triangleq \lambda - \lambda_d$, with λ_d to be defined.

Setting

$$u = u_{\text{PB}} = \dot{\lambda}_d + \frac{R}{k}(1-\theta)\lambda_d + v$$

yields

$$\dot{\tilde{\lambda}} = -\frac{R}{k}(1-\theta)\tilde{\lambda} + v \Rightarrow \dot{H}_d \leq -\alpha\tilde{\lambda}^2 + \tilde{\lambda}v$$

- **Damping injection**: $v = -R_{\text{DI}}\tilde{\lambda}$, with $R_{\text{DI}} > 0$.

(ii) **From flux tracking to force tracking**

The force of magnetical origin F

$$F = \frac{1}{2k}[\lambda_d^2 + \tilde{\lambda}(\tilde{\lambda} + 2\lambda_d)]$$

with $\tilde{\lambda} \to 0$, hence choose λ_d as the solution of

$$F_d = \frac{1}{2k}\lambda_d^2$$

where $F_d > 0$ is some desired force, whose derivative is assumed to be known.

The control becomes

$$u_{\text{PB}} = \sqrt{\frac{k}{2F_d}}\dot{F}_d + R(1-\theta)\sqrt{\frac{2F_d}{k}} + v$$

(iii) **From force tracking to position tracking**

Design $\mathcal{C}_{ol}$ and prove convergence of the position error to zero. Set

$$F_d = m[\ddot{\theta}_* - k_2\dot{\tilde{\theta}} - k_1\tilde{\theta} - k_0\int_0^t \tilde{\theta}(s)ds]$$

This gives the error equation (for mechanical system)

$$\dot{x} = Ax + B\frac{1}{2mk}\tilde{\lambda}(\tilde{\lambda} + 2\lambda_d)$$

with $|\lambda_d| \leq \alpha_1 + \alpha_2|x|$. Stability follows from standard arguments (exponentially stable system perturbed by linearly bounded terms multiplied by an exponential.)

2'. Output–feedback passivity–based control

- One of the main advantages of PBC is the possibility of avoiding state measurements.

- Replace F_d by its **approximate derivative**

$$F_d = m[\ddot{\theta}_* - k_2 \frac{bp}{p+a}\tilde{\theta} - k_1\tilde{\theta} - k_0 \int_0^t \tilde{\theta}(s)ds]$$

with $a,\ b > 0$.

Does not affect the flux tracking property $\tilde{\lambda} \to 0$, nor the control signal, but now u_{PB} can be implemented **feeding–back only** θ **and** $\dot{\theta}$.

The stability analysis mimicks the developments above but with the new A matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_1 & 0 & -k_2 & -k_0 \\ 0 & b & -a & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and the forcing term still satisfies the bound.

3. Integrator backstepping control

- The system is **not in any of the special forms** required for IBC. This does not preclude a backstepping–like design *(Queiroz/Dawson'97)*.

- Start from the mechanical equation and **assuming** F is the control, define an F_d that stabilizes this subsystem. Choose F_d as in FLC and PBC. This yields the first error equation

$$\dot{x} = Ax + \frac{1}{m}B(F - F_d)$$

- Construction of the Lyapunov function. With $V_1 = \frac{1}{2}x^T Px$:

$$\begin{aligned} \dot{V}_1 &= -\frac{1}{2}x^T Qx + \frac{1}{m}x^T PB(F - F_d) \\ &= -\frac{1}{2}x^T Qx + \frac{1}{2mk}x^T PB\tilde{\lambda}(\lambda + \lambda_d) \end{aligned}$$

- Let us now look at the dynamic equation of $\tilde{\lambda}$

$$\dot{\tilde{\lambda}} = -\frac{R}{k}(1-\theta)\lambda + u - \dot{\lambda}_d$$

Decide a control u that stabilizes $\tilde{\lambda}$ plus a term to **compensate for the cross term** in $\dot{V}_1$.

- We have the choice of **linearizing** or **passivizing**.

Take the latter, and set $u = u_{\text{IB}} = u_{\text{PB}}$. We can complete our Lyapunov function with

$$V = V_1 + \frac{1}{\beta} H_d$$

where we have added a tuning parameter $\beta > 0$.

Differentiating V we get

$$\dot{V} \leq -\frac{1}{2} x^T Q x - \frac{\alpha}{\beta} \tilde{\lambda}^2 + \tilde{\lambda}[\frac{1}{\beta} v + \frac{1}{2mc} x^T PB(\lambda + \lambda_d)]$$

The IBC design is completed setting

$$v = -\frac{\beta}{2mk} x^T PB(\lambda + \lambda_d)$$

which removes the cross terms and yields the desired **strict Lyapunov function**

$$\dot{V} \leq -\frac{1}{2} x^T Q x - \frac{\alpha}{\beta} \tilde{\lambda}^2$$

Comparison of the schemes

$$u_{\text{FL}} = \sqrt{\frac{k}{2F}} m v_{\text{FL}}(\theta, \dot{\theta}, F) + R(1-\theta)\sqrt{\frac{2F}{k}}$$

$$u_{\text{PB}} = \sqrt{\frac{k}{2F_d}} m v_{\text{FL}}(\theta, \dot{\theta}, F) + R(1-\theta)\sqrt{\frac{2F_d}{k}}$$

$$u_{\text{IB}} = u_{\text{PB}} - \frac{\beta}{\sqrt{2mk}}[\tilde{\theta},\ \dot{\tilde{\theta}},\ \int_0^t \tilde{\theta}(s)ds] PB(\sqrt{F} + \sqrt{F}_d)$$

where

$$F_d = m[\ddot{\theta}_* - k_2 \frac{bp}{p+a}\tilde{\theta} - k_1\tilde{\theta} - k_0 \int_0^t \tilde{\theta}(s)ds]$$

$$v_{\text{FL}} = \theta_*^{(3)} - k_2[(\frac{1}{m}F - g) - \ddot{\theta}_*] - k_1\dot{\tilde{\theta}} - k_0\tilde{\theta}$$

– Difficult to compare with

$$u_{\text{IDA}} = -K_p(\frac{1}{\alpha}\tilde{\lambda} + \tilde{\theta}) - \left(\frac{\alpha}{m} + K_p\right)\dot{\theta} + R(1-\theta)\sqrt{\frac{2F}{k}}$$

- **Desired** forces: F_d for PBC instead of the **actual** forces F for FLC.
- PBC and IBC are **dynamic**, FLC and IDA are **static**.
- IBC $=$ PBC $+$ cancellation of the cross terms in $\dot{V}$.
- Closed–loop equations for PBC, IDA and IBC are **nonlinear**. Tuning?
- In PBC we get a **cascade** structure, in IBC we get

$$\dot{z} = (A_{\rm DI} + A_{\rm SK})z,$$

where $A_{\rm DI}$ is “stable” and $A_{\rm SK}$ satisfies a **skew–symmetry** property. Similar property for IDA.

- There are no **output feedback** versions of IBC or IDA. However, if we want to **add damping** to the PBC we need the full state.

Simulation results

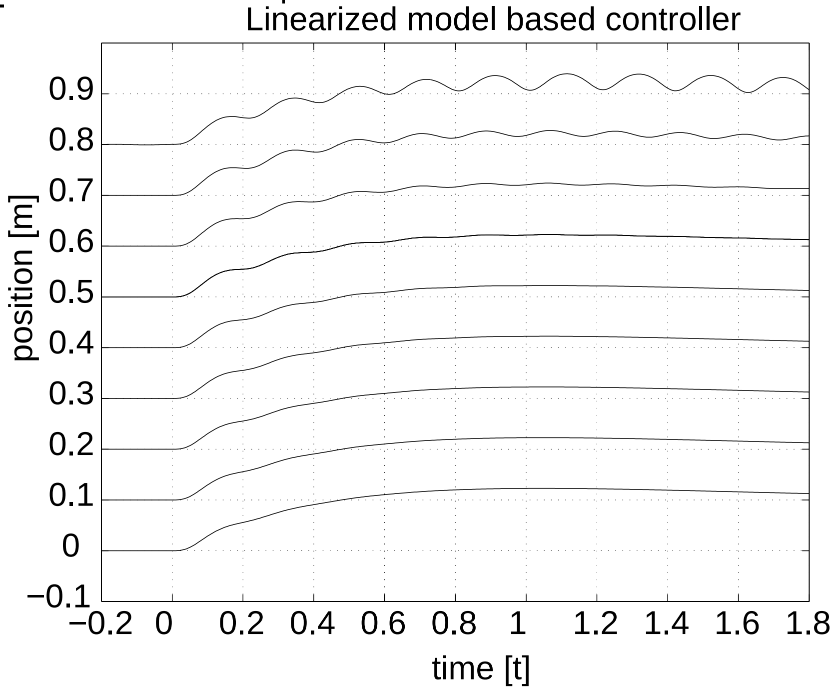

Figure 13: Linear controller, $D(p) = (p + 10)^3$.

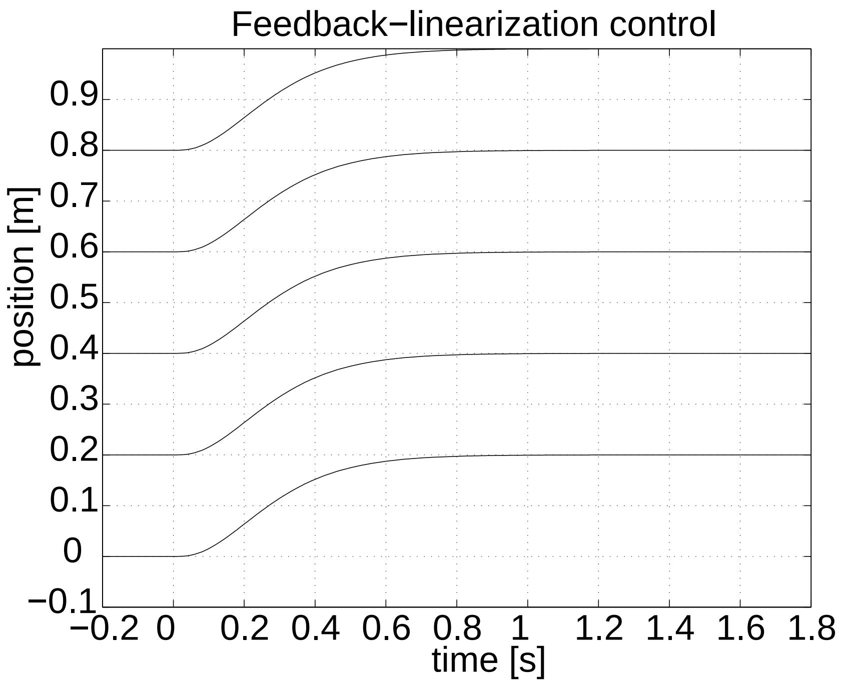

Figure 14: Feedback linearization, $D(p) = (p + 10)^3$.

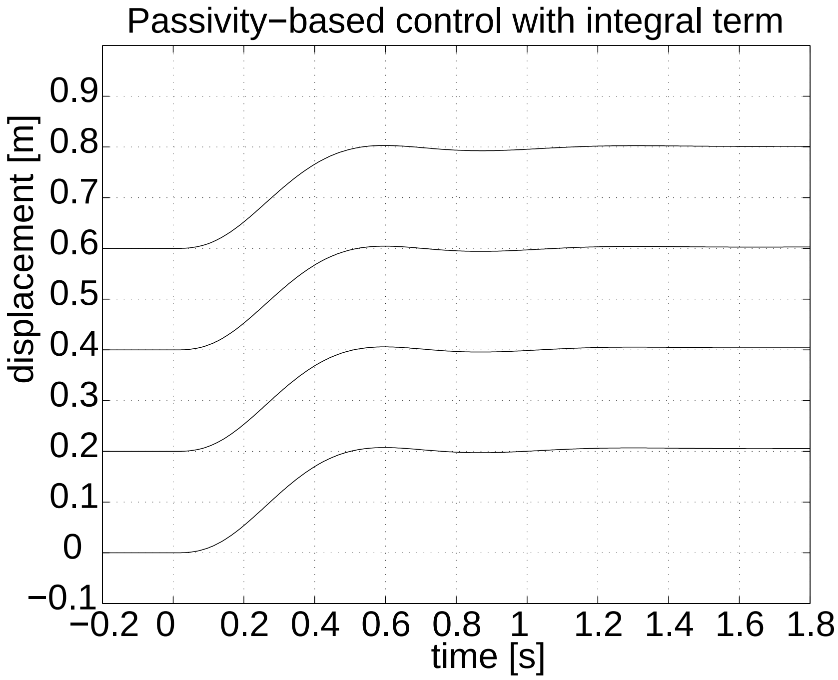

Figure 15: PBC with integrator.

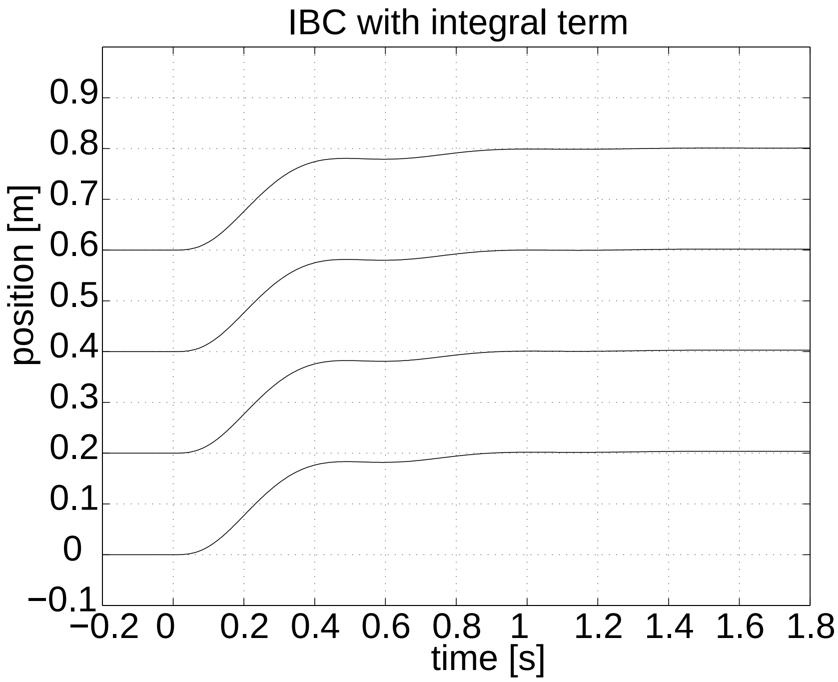

Figure 16: Backstepping control, $\beta = 100$.

Conclusions

- IDA is, by far the simplest.

- “Which storage–Lyapunov function you (can) want to assign?” In Standard PBC: a quadratic in increments, in IDA the **energy function**. In IBC, for cascaded systems, the Lyapunov function is **recursively** constructed. IBC can be profitably combined with PBC at this stage.

- Avoiding cancellation of nonlinearities enhances the robustness of the scheme? Unquestionably established, theoretically and experimentally, in *(Kim, et al’96)*. Not possible in this **simple** example.

♡ Establish some common framework to compare robustness and **performance**. Assess the degrees of freedom provided to the designer.

♡ Skew–symmetry of IBC ≈ workless–forces ≈ **energy transformation**.

7.2 Boost converter

- Model (under fast switching), $x(0) \in \mathcal{R}^2_{>0}$

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & -u \\ u & \frac{-1}{R} \end{bmatrix}}_{J(u)} \frac{\partial H}{\partial x}(x) + \begin{bmatrix} E \\ 0 \end{bmatrix}$$

- **Control objective**: regulate $\frac{1}{C}x_2$ to a desired constant value $V_* > E$, verifying

C.1 Only x_2 measurable.

C.2 $u \in (0, 1)$.

C.3 $x \in \mathcal{R}^2_{>0}$.

C.4 R is unknown.

- *Main contribution:* Stabilization via IDA–PBC with a simple static nonlinear output feedback.

Proposition

For all $R > 0$ the IDA–PBC

$$u = u_* \left(\frac{x_2}{x_{2*}} \right)^{\alpha}, \quad 0 < \alpha < 1$$

yields

(i) $x_* = (\frac{L}{RE} V_*^2,\ C V_*)$ is *asymptotically stable* with Lyapunov function

$$H_d(x) = \frac{1}{2L} x_1^2 + \frac{1}{2C} x_2^2 + \kappa_1 x_2^{2(1-\alpha)} - (\kappa_2 + \kappa_3 x_1) x_2^{1-\alpha}$$

(ii) Domain of attraction:

$$\Xi_\alpha \stackrel{\triangle}{=} \{x | x \in \mathcal{R}^2_{>0} \quad \text{and} \quad H_d(x) \leq H_d(0, x_{2*})\}$$

is such that

$$\boxed{x(0) \in \Xi_\alpha \Rightarrow x(t) \in \Xi_\alpha \text{ and } \lim_{t \to \infty} x(t) = x_*}$$

(iii) Saturation: $\forall x_*,\ \exists \alpha \in (0, 1)$ s.t.

$$\boxed{x(0) \in \Xi_\alpha \Rightarrow 0 < u(t) < 1.}$$

Proof

- **IDA** Select

$\mathcal{R}_a = diag\{R_a, -\frac{1}{R}\} \Rightarrow \mathcal{R}_d = diag\{R_a, 0\}$.

- **Integrability** Key PDE[a]

$$K \triangleq \frac{\partial H_a}{\partial x}(x) = \frac{1}{\beta(x_2)} \begin{bmatrix} -\frac{1}{RC}x_2 \\ -\frac{1}{L}R_a x_1 - E + \frac{R_a}{RC}\frac{x_2}{\beta(x_2)} \end{bmatrix}$$

PDE solvable $\Leftrightarrow \frac{\partial K_2}{\partial x_1}(x) = \frac{\partial K_1}{\partial x_2}(x) \Leftrightarrow$

$$\frac{d\beta}{dx_2}(x_2) = \frac{\alpha}{x_2}\beta(x_2)$$

where $\alpha \triangleq 1 - \frac{R_a RC}{L}$. Thus,

$$u = c_1 x_2^{\alpha}$$

- **Equilibrium assignment**: c_1 such that

$$\frac{\partial H_d}{\partial x}(x_*) = \frac{\partial H}{\partial x}(x_*) + \frac{\partial H_a}{\partial x}(x_*) = 0$$

This yields $c_1 = \frac{u_*}{x_{2*}^{\alpha}}$.

- **Hessian condition**

$$\frac{\partial^2 H_d}{\partial x^2}(x_*) = \begin{bmatrix} \frac{1}{L} & -\frac{R_a}{u_* L} \\ -\frac{R_a}{u_* L} & \frac{1}{C} + \frac{(R_a x_{1*} + EL)\alpha}{u_* L x_{2*}} + \frac{(1-2\alpha)R_a}{u^{*2}RC} \end{bmatrix}$$

Positive definite $\Leftrightarrow -1 < \alpha < 1$.

[a] Assuming $J(\beta(x)) - \mathcal{R}_d$ is invertible.

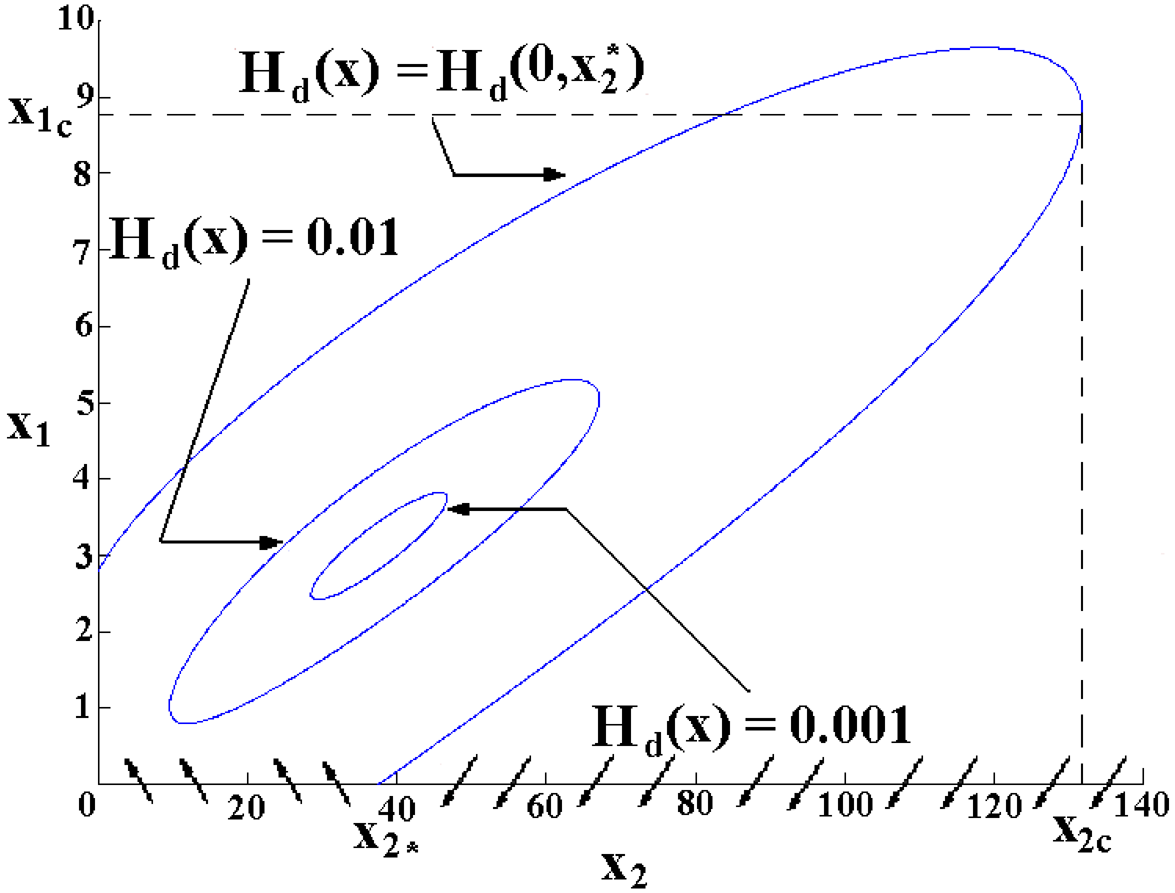

Figure 17: Estimated domain of attraction

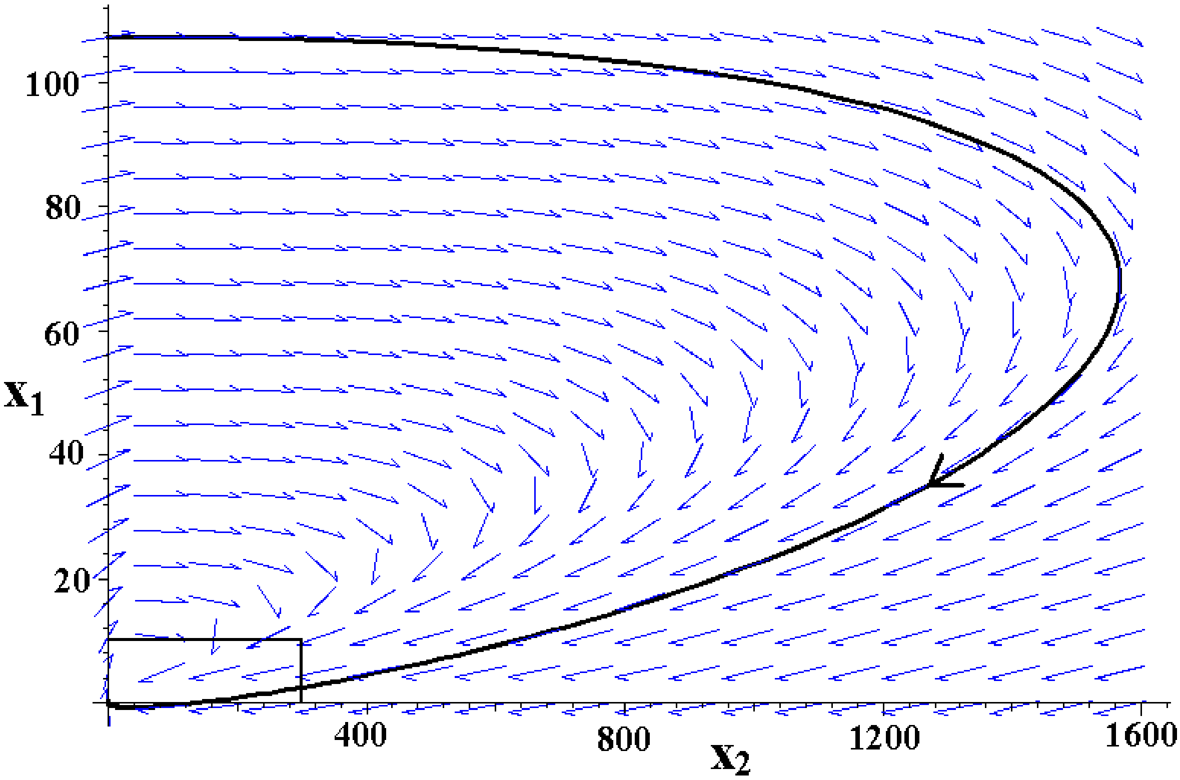

Figure 18: Exact domain of attraction

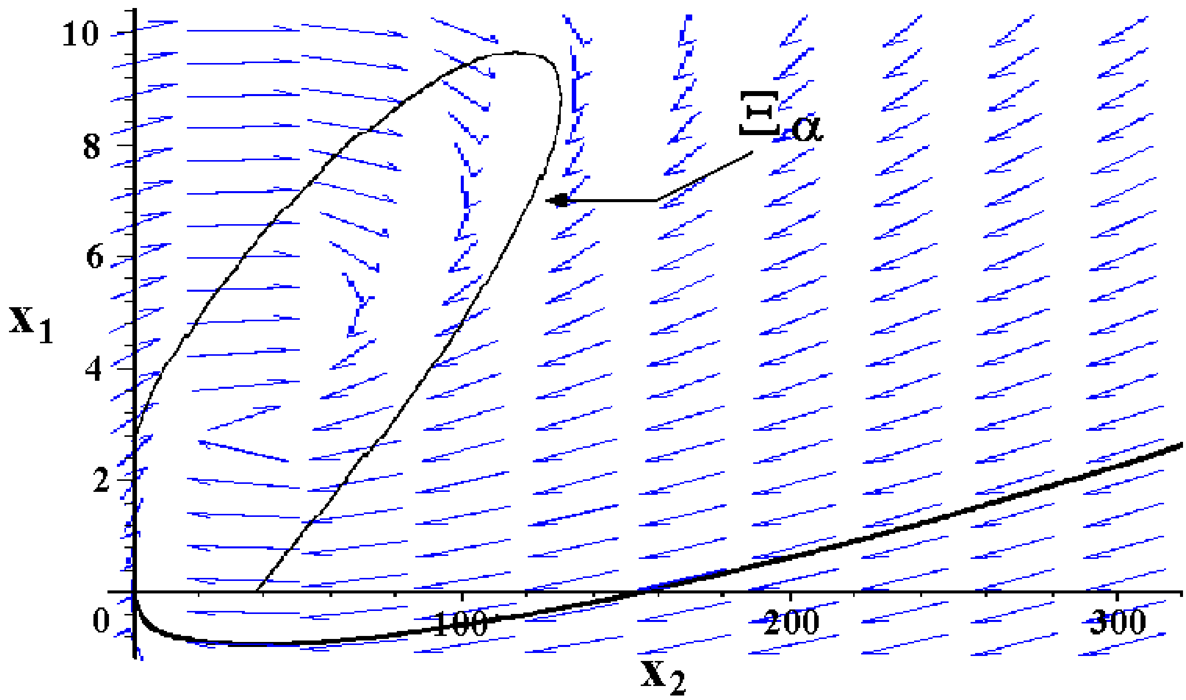

Figure 19: Zoom around the equilibrium point and set Ξ_α.

Comparison with the Standard PBC.

(1) The model of the system is written as

$$M\dot{z} + J(u)z + \mathcal{R}z = g$$

where $z \in I\!R^2$, $M := diag\{L,\ C\}, g = [E, 0]^\top$.

(2) An *implicit* definition of the controller is derived from a copy of the system with additional damping as

$$M\dot{z}_d + J(u)z_d + \mathcal{R}z_d = g + \mathcal{R}_{di}\tilde{z}$$

where $\mathcal{R}_{di} := diag\{R_1,\ 0\},\ R_1 > 0$, $z_d \in I\!R^2$ is an auxiliary vector, $\tilde{z} := z - z_d$, and z_d will be defined later. The idea is that, for all u, the error equation

$$M\dot{\tilde{z}} + [J(u) + \mathcal{R}_d]\tilde{z} = 0$$

with $\mathcal{R}_d := \mathcal{R} + \mathcal{R}_{di}$, is exponentially convergent, that is, $\tilde{z} \to 0$ (exp.).[a]

(3) Find a control u such that $\tilde{z} \to 0 \Rightarrow z_2 \to V_d$.

- If we set $z_{2d} = V_d$, solve for u and define a differential equation for z_{1d}, leads to an internally *unstable* system, because the zero dynamics is unstable, and the controller implements an (asymptotic) inversion.

---

[a] Easily established evaluating the derivative of $\tilde{z}^\top M\tilde{z}$.

(4) Fix instead z_{1d} to its desired value $\frac{V_d^2}{RE}$. This leads to

$$\begin{aligned} C\dot{z}_{2d} &= -\frac{1}{R_L} z_{2d} + \frac{V_d^2}{R_L E z_{2d}}(E + R_1 \tilde{z}_1) \\ u &= \frac{1}{z_{2d}}(E + R_1 \tilde{z}_1) \end{aligned}$$

z_{2d} is the state of our dynamic controller.

(5) To complete the stability analysis we must show that z_{2d} remains bounded, which follows from the minimum phase properties of the system with output z_1.

Comparing the solutions

(i) Obvious complexity reduction.

(ii) IDA is more "natural": no enforcement of quadratic storage functions.

(iii) No need for stable invertibility.[a]

[a] Brings along robustness problems inherent to linearization.

7.3 Mechanical systems

A. IDA–PBC theory

- To stabilize some underactuated mechanical devices it is necessary to modify the *total* energy function. In open loop

$$H(q,p) = \frac{1}{2} p^\top M^{-1}(q) p + V(q)$$

where $q \in \mathbb{R}^n$, $p \in \mathbb{R}^n$ are the generalized position and momenta, respectively, $M(q) = M^\top(q) > 0$ is the inertia matrix, and $V(q)$ is the potential energy.

- Model

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} \nabla_q H \\ \nabla_p H \end{bmatrix} + \begin{bmatrix} 0 \\ G(q) \end{bmatrix} u$$

Control $u \in \mathbb{R}^m$, and assume $\text{rank}(G) = m < n$.

Convenient to decompose $u = u_{es}(q,p) + u_{di}(q,p)$

- **Target Dynamics**

Desired (closed loop) energy function

$$H_d(q,p) = \frac{1}{2} p^\top M_d^{-1}(q) p + V_d(q)$$

where $M_d = M_d^\top > 0$ and $V_d(q)$

$$q_* = \arg\min V_d(q)$$

Thus,

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = [J_d(q,p) - R_d(q,p)] \begin{bmatrix} \nabla_q H_d \\ \nabla_p H_d \end{bmatrix}$$

where

$$\begin{aligned} J_d &= -J_d^\top = \begin{bmatrix} 0 & M^{-1}M_d \\ -M_d M^{-1} & J_2(q,p) \end{bmatrix} \\ R_d &= R_d^\top = \begin{bmatrix} 0 & 0 \\ 0 & GK_vG^\top \end{bmatrix} \geq 0 \end{aligned}$$

- **Observations**:
  - The $(1,2)$–block of J_d is determined from $\dot{q} = M^{-1}p$.
  - The matrix R_d is included to add damping

    $$u_{di} = -K_v G^\top \nabla_p H_d$$

    where we take $K_v = K_v^\top > 0$. This explains the $(2,2)$–block of R_d.
  - J_2[a] can be used as **free parameters**.

[a] Also, some of the elements of M_d.

Energy Shaping

- Equate the open and closed–loop dynamics

$$Gu_{es} = \nabla_q H - M_d M^{-1} \nabla_q H_d + J_2 M_d^{-1} p$$

This leads to

$$G^{\perp}\{\nabla_q H - M_d M^{-1} \nabla_q H_d + J_2 M_d^{-1} p\} = 0$$

where $G^{\perp} G = 0$. Is a set of nonlinear PDE's with unknowns M_d and V_d, with J_2 a *free* parameter, and p an independent coordinate.

- Control law u_{es} is given as

$$u_{es} = (G^{\top} G)^{-1} G^{\top} (\nabla_q H - M_d M^{-1} \nabla_q H_d + J_2 M_d^{-1} p)$$

- PDE's naturally separated into

$$\begin{aligned} 0 &= G^{\perp}\left\{\nabla_q (p^{\top} M^{-1} p) - M_d M^{-1} \nabla_q (p^{\top} M_d^{-1} p) + 2 J_2 M_d^{-1} p\right\} \\ 0 &= G^{\perp}\{\nabla_q V - M_d M^{-1} \nabla_q V_d\} \end{aligned}$$

The first equation is a nonlinear PDE, the second is a simple linear PDE.

- **More explicit form** Using

$$\nabla_q(z^\top P(q)z) = [\nabla_q(P(q)z)]^\top z$$

which holds $\forall z \in I\!R^n, P = P^\top \in I\!R^{n\times n}$, yields

$$G^\perp \left\{ \left[[\nabla_q(M^{-1}p)]^\top - M_d M^{-1}[\nabla_q(M_d^{-1}p)]^\top + 2J_2 M_d^{-1} \right] p \right\} = 0$$

Then, we apply the identity

$$\nabla_q(P(q)z) = \sum_{k=1}^{n} \nabla_q(P_{(\cdot,k)})z_k$$

where $P_{(\cdot,k)}$ denotes the k–th column of the matrix P, reparametrize J_2, in terms of the matrices $U_k(q) = -U_k^\top(q) \in I\!R^{n\times n}$, as

$$2J_2 = \sum_{k=1}^{n} U_k p_k$$

and equate terms in p_k to obtain

$$G^\perp \left\{ \left[\nabla_q(M^{-1}_{(\cdot,k)}) \right]^\top - M_d M^{-1} \left[\nabla_q(M_d^{-1})_{(\cdot,k)} \right]^\top + U_k M_d^{-1} \right\} = 0$$

where U_k are *designer chosen* matrices.

- **Key idea:** choose U_k to solve PDE's with $M_d(q) = M_d^\top(q) > 0$ and

$$q_* = \arg\min V_d(q)$$

Remarks

– Characterize a class of underactuated mechanical systems for which IDA–PBC yields smooth stabilization, in terms of solvability of the PDE's.

– Two "extreme" particular cases:

(i) If we do not modify the interconnection and inertia matrices then we recover the well–known potential energy shaping procedure of PBC. Indeed, if $M_d = M$ and $J_2 = 0$, then

$$u_{es} = (G^\top G)^{-1} G^\top (\nabla_q V - \nabla_q V_d)$$

(ii) If we do not change the potential energy, but only modify the kinetic energy, and **fix**

$$J_2(q,p) = M_d M^{-1} \left\{ [\nabla_q (M M_d^{-1} p)]^\top - \nabla_q (M M_d^{-1} p) \right\} M^{-1} M_d$$

we recover the *controlled–Lagrangian* method.

Warning M_d should be*negative definite*:[a] hard to justify from a physical viewpoint!

[a]Relative equilibria, i.e., $(x = \bar{x}, \dot{x} = 0, \dot{y} = 0)$, of *conservative* systems are stable if the Hessian of the total energy is *definite* (either positive or negative).

B. Controlled–Lagrangian method

- Starting from an EL system

$$\frac{d}{dt}\nabla_{\dot q}L(q,\dot q)-\nabla_q L(q,\dot q)=G(q)u,\ L=\frac{1}{2}\dot q^\top M(q)\dot q-V(q)$$

we want the closed–loop

$$\frac{d}{dt}\nabla_{\dot q}L_c(q,\dot q)-\nabla_q L_c(q,\dot q)=0, L_c=\frac{1}{2}\dot q^\top M_c(q)\dot q-V_c(q)$$

- Achievable Lagrangians L_c characterized by

$$\begin{aligned} 0 &= G^\perp\{[\nabla_q(M\dot q)-MM_c^{-1}\nabla_q(M_c\dot q)]\dot q - \\ &- \frac{1}{2}[\nabla_q(\dot q^\top M\dot q)-MM_c^{-1}\nabla_q(\dot q^\top M_c\dot q)]\} \\ 0 &= G^\perp\{\nabla_q V - MM_c^{-1}\nabla_q V_c\} \end{aligned}$$

- **Exactly coincides** with our PDE's, with

$$M_c(q)\stackrel{\triangle}{=}MM_d^{-1}M$$

- We can add a term

$$M_d M^{-1}\{[\nabla_q Q(q)]^\top-\nabla_q Q(q)\}M^{-1}M_d$$

with arbitrary $Q(q)$, preserving EL structure. This corresponds to (i.e., *intrinsic* gyroscopic terms)

$$L_c=\frac{1}{2}\dot q^\top M_c\dot q+\dot q^\top Q-V_c$$

C. Solving the Matching PDE's

- If $m = n - 1$, and M depends only on the unactuated coordinate, then the nonlinear PDE can be transformed, with a suitable choice of M_d and J_2, into a set of *ordinary differential equations*.

- *(Auckley, et al., '2000)* showed that all of the solutions of the controlled Lagrangian PDE's may be obtained by sequentially solving two sets of first order *linear* PDE's.

- *(Bloch, et al., '2000)* have proven that, in some cases which includes some well–known examples, the solution of these PDE's can actually be obviated, restricting M_c.

D. Inertia Wheel Pendulum

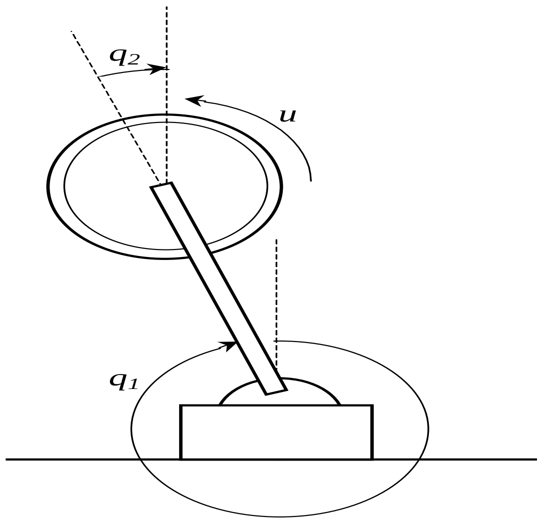

Figure 20:

- Model

$$\begin{bmatrix} I_1 + I_2 & I_2 \\ I_2 & I_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} -mgl\sin(\theta_1) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

- $V = V(\theta_1)$, but passive output $\dot{\theta}_2$. Hence, *not stabilizable* with potential energy shaping!

- Change of coordinates $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \theta_1 + \theta_2 \end{bmatrix}$, leads to PCH model with $p = [I_1\dot{q}_1, I_2\dot{q}_2]^\top$,

$$H(q,p) = \frac{1}{2}p^\top M^{-1}p + m_3(\cos q_1 - 1)$$

where $m_3 \stackrel{\triangle}{=} mgl$, $M = \text{diag}\{I_1, I_2\}$, and $G = [-1, 1]^\top$.

Energy Shaping

M is independent of q, hence, we can take $J_2 = 0$ and M_d to be a constant matrix too

$$M_d = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} a_1 > 0,\ a_1 a_3 > a_2^2$$

The only PDE to be solved is then

$$\left(\frac{a_1 + a_2}{I_1}\right) \frac{\partial V_d}{\partial q_1} + \left(\frac{a_2 + a_3}{I_2}\right) \frac{\partial V_d}{\partial q_2} = -m_3 \sin(q_1)$$

whose general solution is

$$V_d(q) = \frac{I_1 m_3}{a_1 + a_2} \cos(q_1) + \Phi\left(q_2 + \gamma_2 q_1\right)$$

where Φ is an arbitrary differentiable function,

$$\gamma_2 \triangleq -I_1(a_2 + a_3)/(I_2(a_1 + a_2))$$

Now,

$$\nabla_q V_d(0) = 0 \quad \Leftrightarrow \quad \nabla \Phi(0) = 0$$
$$\nabla_q^2 V_d(0) > 0 \quad \Leftarrow \quad \nabla^2 \Phi(0) = 0,\ a_2 < -a_1$$

Simple choice

$$\Phi = \frac{k_1}{2}\left(q_2 + \gamma_2 q_1\right)^2, k_1 > 0$$

Yielding

$$u_{es} = \gamma_1 \sin(q_1) + k_p(q_2 + \gamma_2 q_1)$$

where we have defined

$$\gamma_1 \triangleq \frac{a_2}{a_1 + a_2} m_3, k_p \triangleq -k_1 \left[\frac{a_1 a_3 - a_2^2}{I_2(a_1 + a_2)}\right] > 0$$

Finally, *admissible region* for the tuning gains

$$M_d > 0 \Leftrightarrow \gamma_1 > m_3, \gamma_2 > \frac{I_1}{I_2} \frac{\gamma_1}{\gamma_1 - m_3}$$

Damping Injection and Stability Analysis

- Add damping feeding back the new passive output

$$G^\top \nabla_p H_d = k_2(\dot{q}_2 + \gamma_2 \dot{q}_1)$$

where $k_2 \stackrel{\triangle}{=} -\frac{I_2(a_1+a_2)}{a_1 a_3 - a_2^2} > 0$.

Proposition

The static state–feedback IDA–PBC

$$u = \gamma_1 \sin(q_1) + k_p(q_2 + \gamma_2 q_1) - k_v(\dot{q}_2 + \gamma_2 \dot{q}_1)$$

ensures

$$\lim_{t\to\infty} (q(t), p(t)) = (2k\pi, -2j\pi, 0, 0), \; k, j \in I\!N$$

for all initial conditions—except a set of zero measure.

Output Feedback

Obviate velocity measurement feeding back the dirty derivative of positions

$$u_{di} = -\frac{k_v p}{\tau p + 1}(q_2 + \gamma_2 q_1)$$

Proposition

Dynamic output feedback IDA–PBC

$$u = -\gamma_1 \sin(q_1) + k_p(q_2 + \gamma_2 q_1) + u_{di}$$

$$\begin{aligned} \dot{z} &= -\frac{1}{\tau} z + \frac{k_v}{\tau^2}(q_2 + \gamma_2 q_1) \\ u_{di} &= z - \frac{k_v}{\tau}(q_2 + \gamma_2 q_1) \end{aligned}$$

Proof

$$\dot{u}_{di} = -\frac{1}{\tau} u_{di} - \frac{k_v}{k_2 \tau} G^\top \nabla_p H_d$$

Consequently, with $W(q, p, u_{di}) \triangleq H_d + \frac{k_v}{2k_2\tau} u_{di}^2$,

$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{u}_{di} \end{bmatrix} = \begin{bmatrix} J_d & \frac{k_v}{k_2\tau} G \\ -\frac{k_v}{k_2\tau} G^\top & -\frac{k_v}{k_2\tau^2} \end{bmatrix} \begin{bmatrix} \nabla_q W \\ \nabla_p W \\ \nabla_{u_{di}} W \end{bmatrix}$$

with $\dot{W} = -\frac{k_2}{k_v}(u_{di})^2$.

Simulation Results

- Parameters: $I_1 = .1, I_2 = .2, m_3 = 10,$
$\gamma_1 = 30,\ \gamma_2 = 0.2.$

- IC's: $q(0) = (3, 0)$.

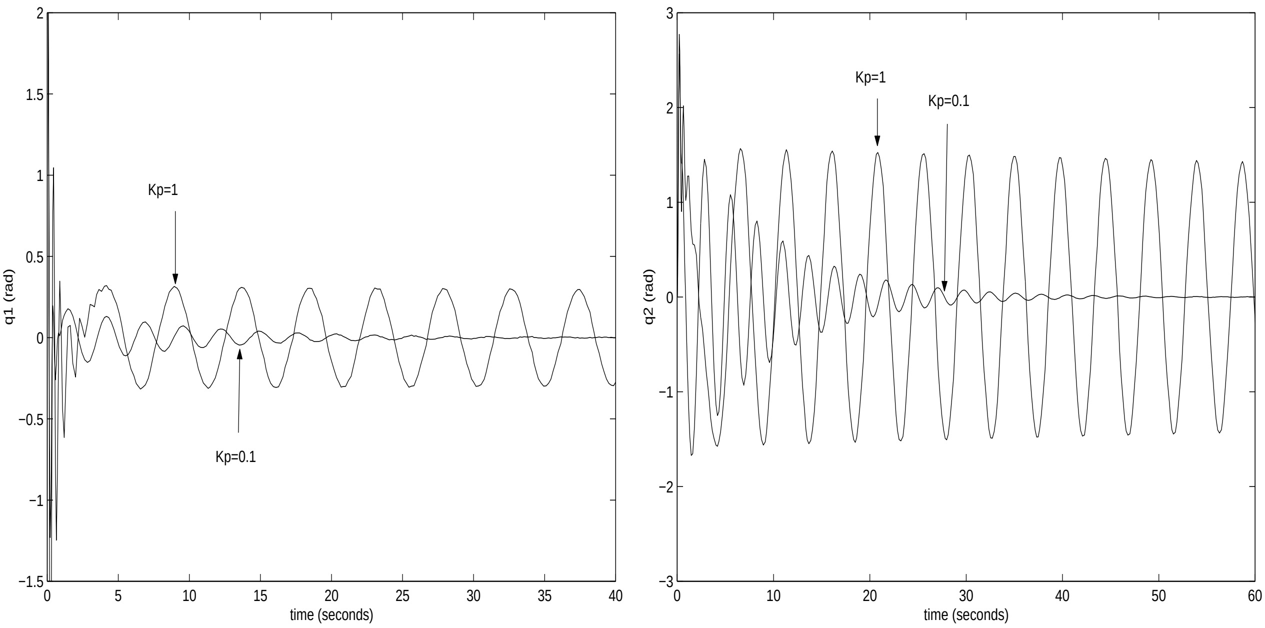

Figure 21: Evolution of $q_1(t)$ (left) and $q_2(t)$ for different values of k_p, letting $k_v = 10$

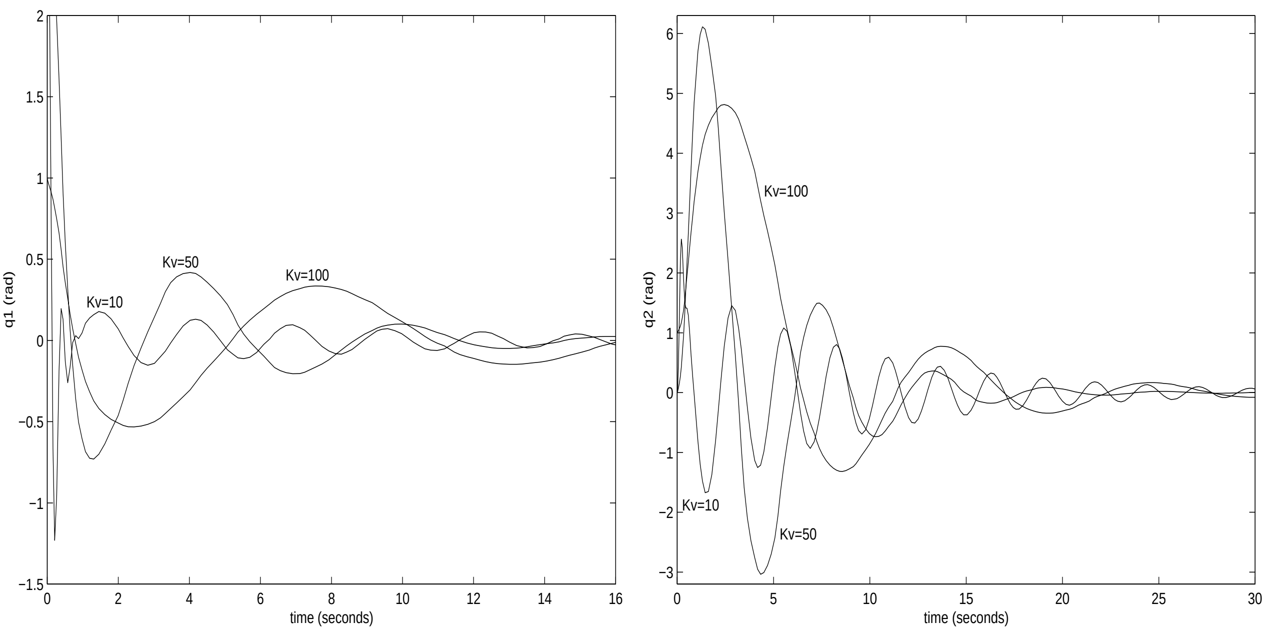

Figure 22: Evolution of $q_1(t)$ (left) and $q_2(t)$ for different values of k_v, and $k_p = 0.1$

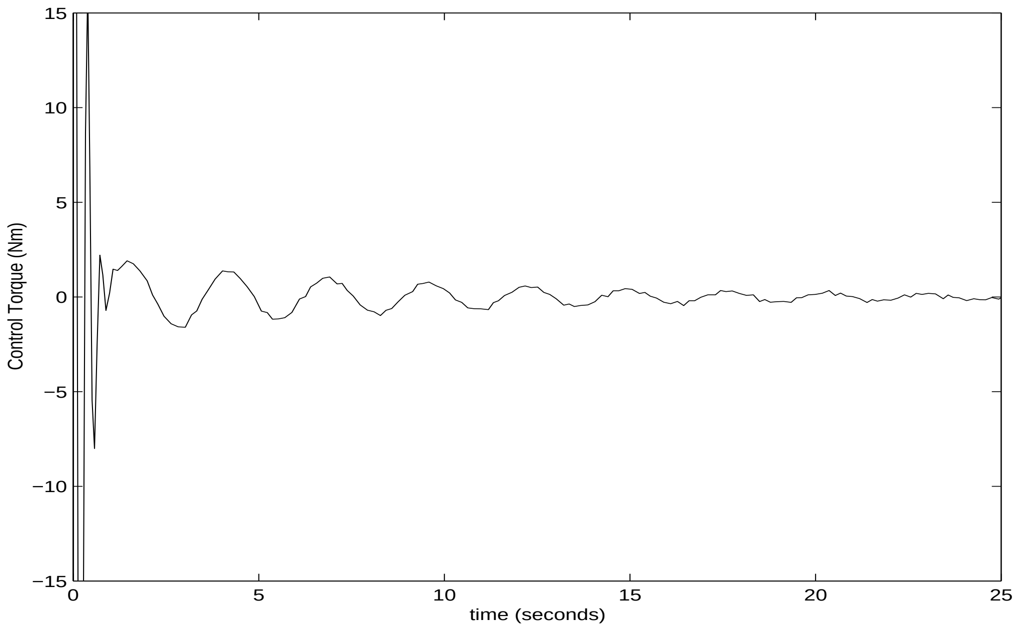

Figure 23: Control signal for $k_v = 10$, and $k_p = 0.1$ (Nm)

E. Ball and Beam

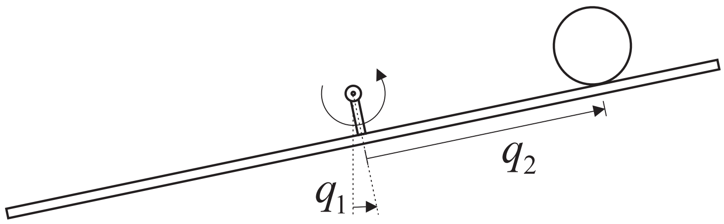

Figure 24:

- Model

$$\begin{aligned} m_2\ddot{q}_1 + m_3\sin(q_2) - m_1 q_1 \dot{q}_2^2 &= 0 \\ (1+m_1 q_1^2)\ddot{q}_2 + 2m_1 q_1 \dot{q}_1 \dot{q}_2 + m_3 q_1 \cos(q_2) &= u \end{aligned}$$

- Energy

$$H = \frac{1}{2}\dot{q}^\top \begin{bmatrix} m_2 & 0 \\ 0 & 1+m_1 q_1^2 \end{bmatrix} \dot{q} + m_3 q_1 \sin(q_2)$$

- $M = M(q_1)$, hence $M_d = \begin{bmatrix} a_1(q_1) & a_2(q_1) \\ a_2(q_1) & a_3(q_1) \end{bmatrix}$

Kinetic energy

- With a suitable $J_2(q,p)$, the PDE reduces to the ODEs

$$
\begin{aligned}
\frac{d}{dq_1} a_1(q_1) &= \frac{2m_1 m_2 q_1}{(1+m_1 q_1^2)^2} \frac{a_2^2}{a_1} \\
\frac{d}{dq_1} a_2(q_1) &= \frac{2m_1 m_2 q_1}{(1+m_1 q_1^2)^2} \frac{a_2 a_3}{a_1}
\end{aligned}
$$

to be solved for a_1, a_2, and a_3 is a "free" parameter.

- Fix $a_3 = \gamma a_2 a_1$, with $\gamma \neq 0$. Yielding

$$
a_1 = \sqrt{\frac{2m_1}{\gamma^2 m_2} q_1^2 + C} \qquad a_2 = \frac{1}{\gamma m_2}(1 + m_1 q_1^2)
$$

with C a free integration constant.

Potential energy

- PDE

$$
a\sqrt{1+{x_1}^2}\frac{\partial V_d}{\partial x_1}(x) + b\frac{\partial V_d}{\partial x_2}(x) = \sin(x_2)
$$

where $x_1 = \sqrt{m_1} q_1,\ x_2 = q_2$. Maple solution

$$
V_d(x) = -\frac{1}{b}\cos(x_2) + \Phi\left(\frac{b}{a}\text{arcsinh}(x_1) - x_2\right)
$$

with Φ taken, again, quadratic.

Proposition

The IDA–PBC

$$u_{es} = -\frac{q_1}{1+m_1 q_1^2}[b_1(1+m_1 q_1^2)^{\frac{1}{2}} p_1^2 +$$

$$+b_2 p_1 p_2 + b_3(1+m_1 q_1^2)^{-\frac{1}{2}} p_2^2] + \xi(q)$$

with

$$\xi(q) = \alpha_1 q_1 \cos q_2 + (1+m_1 q_1^2)^{\frac{1}{2}}[\alpha_2 \sin q_2 + k_p(\alpha_3 q_2 +$$

$$+\alpha_4 \text{arcsinh}(\sqrt{m_1} q_1))]$$

and, damping injection

$$u_{di} = -k_v \frac{m_2}{1+m_1 q_1^2}\left(-p_1 + \sqrt{2m_2}\frac{p_2}{\sqrt{1+m_1 q_1^2}}\right)$$

ensures asymptotic stability of the origin and $\lim_{t\to\infty} q_1(t) = 0$ for all initial conditions, except a set of zero measure.

Simulations

- Parameters: $m_1 = m_2 = m_3 = 1$ and $\gamma = k_p = 1$,
- IC's $q_1(0) = 1,\ q_2(0) = 1$

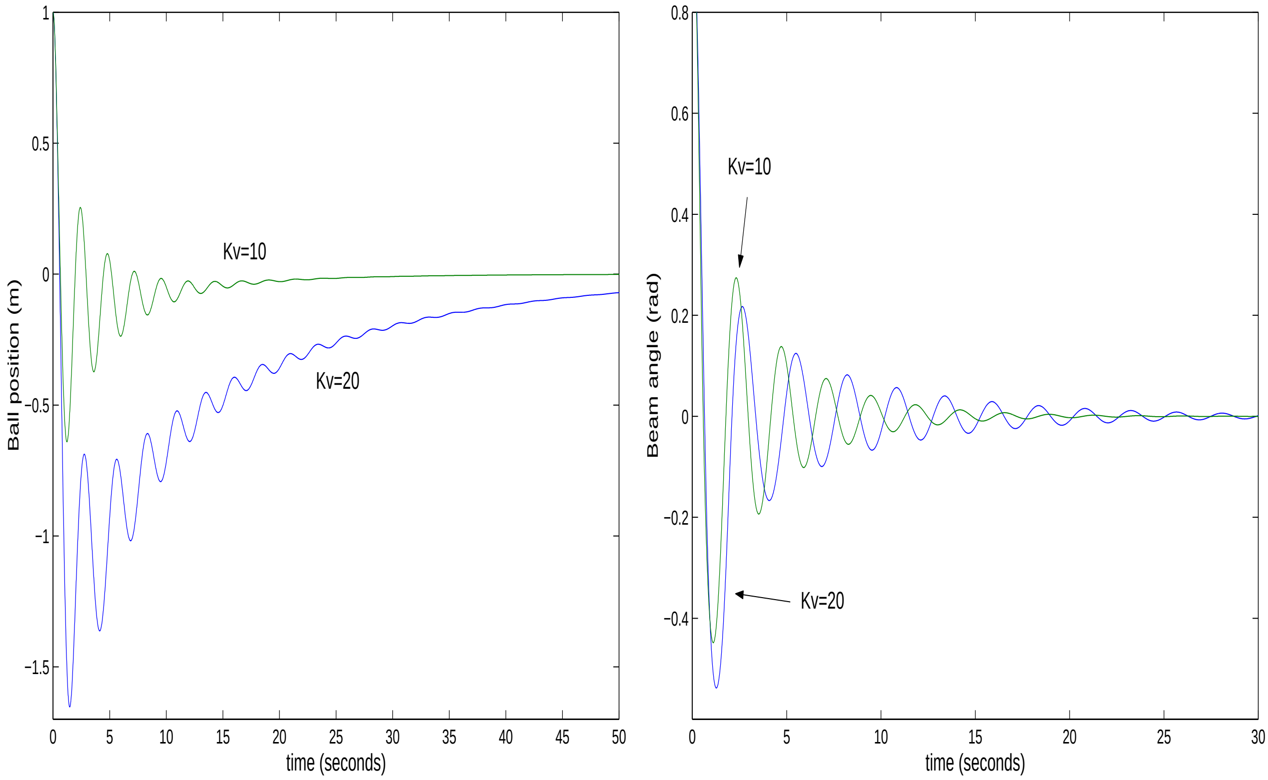

Figure 25: Positions of the ball (m) (left) and the beam (rads) with different damping constants

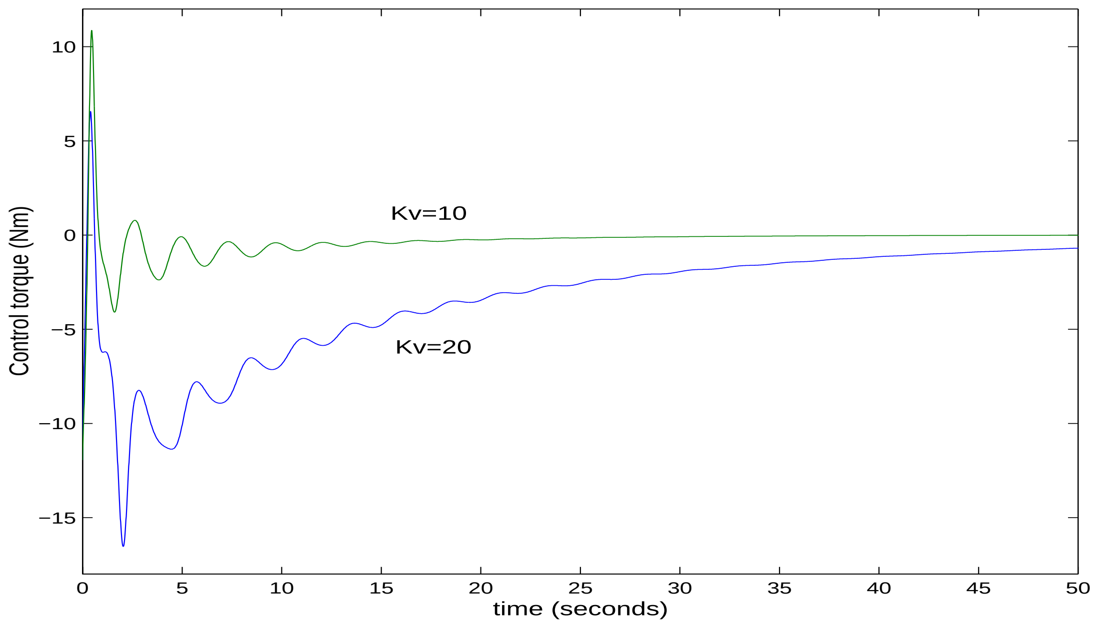

Figure 26: Control signal (Nm)

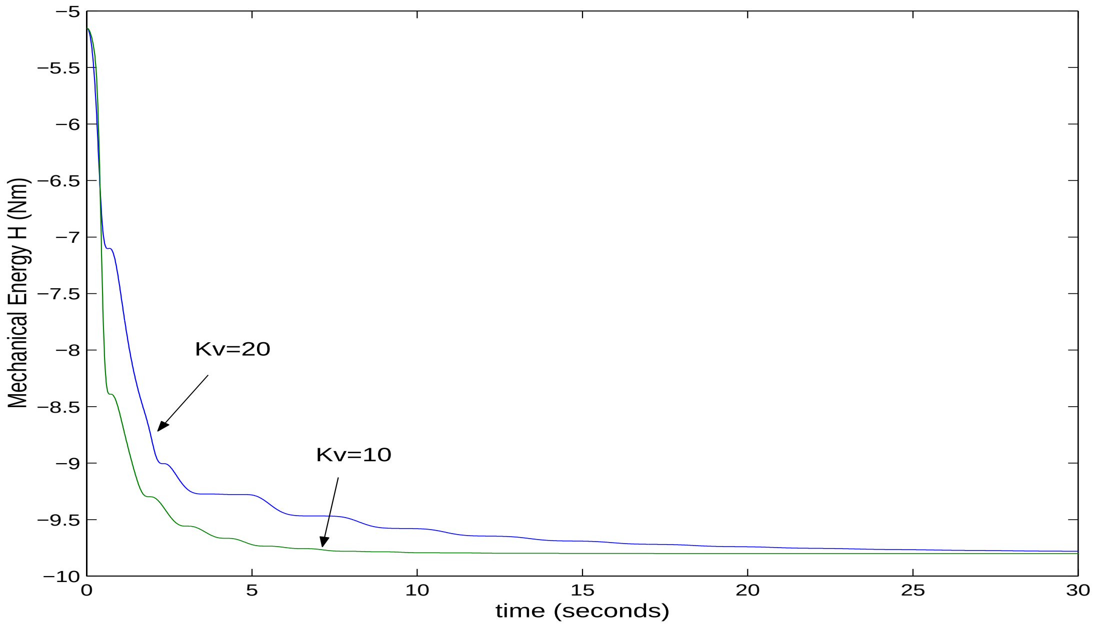

Figure 27: Total energy function $t \mapsto H_d$ (Nm)

7.4 PM Synchronous Motor

A. Model

- dq model

$$
\begin{aligned}
L_d \frac{di_d}{dt} &= -R_s i_d + \omega L_q i_q + v_d \\
L_q \frac{di_q}{dt} &= -R_s i_q - \omega L_d i_d - \omega \Phi + v_q \\
J \frac{d\omega}{dt} &= P \cdot ((L_d - L_q) i_d i_q + \Phi i_q) - \tau_l
\end{aligned}
$$

ω is angular velocity, v_d, v_q, i_d, i_q are voltages and currents. P is the number of pole pairs, L_d and L_q are stator inductances, R_s is stator winding resistance, τ_l is a constant *unknown* load torque, and Φ and J are the dq back emf constant and the moment of inertia.

- Energy function

$$H(x) = \frac{1}{2}\left(L_d i_d^2 + L_q i_q^2 + \frac{J}{P}\omega^2\right) = \frac{1}{2} x^\top \mathbf{D}^{-1} x$$

where

$$x = \mathbf{D} \begin{bmatrix} i_d \\ i_q \\ \omega \end{bmatrix}, \quad \mathbf{D} \triangleq \begin{bmatrix} L_d & 0 & 0 \\ 0 & L_q & 0 \\ 0 & 0 & \frac{J}{P} \end{bmatrix}$$

PCH model

$$\dot{x} = [\mathbf{J}(x) - \mathcal{R}(x)]\frac{\partial H}{\partial x}(x) + g(x)u + \zeta$$

with

$$g = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},\ u = \begin{bmatrix} v_d \\ v_q \end{bmatrix},\ \zeta = \begin{bmatrix} 0 \\ 0 \\ -\frac{\tau_l}{P} \end{bmatrix}$$

$\mathcal{R} = \mathit{diag}\{R_s, R_s, 0\}$ and

$$\mathbf{J}(x) = \begin{bmatrix} 0 & 0 & x_2 \\ 0 & 0 & -(x_1 + \Phi) \\ -x_2 & x_1 + \Phi & 0 \end{bmatrix}$$

Desired equilibrium ("maximum torque per ampere" principle)

$$x_* = \begin{bmatrix} x_{1*} \\ x_{2*} \\ x_{3*} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{L_q \tau_l}{P\Phi} \\ \frac{J}{P}\omega_* \end{bmatrix}$$

B. Natural Interconnection Controller

Without modification, that is, $\mathbf{J}_a(x) = \mathcal{R}_a(x) = 0$.

$$v_d = -R_s \frac{\partial H_a}{\partial x_1} + x_2 \frac{\partial H_a}{\partial x_3}$$

$$v_q = -R_s \frac{\partial H_a}{\partial x_2} - (x_1 + \Phi) \frac{\partial H_a}{\partial x_3}$$

$$-x_2 \frac{\partial H_a}{\partial x_1} + (x_1 + \Phi) \frac{\partial H_a}{\partial x_2} = -\frac{1}{P}\tau_l$$

Solution is

$$H_a(x) = \frac{\tau_l}{P} \arctan(\frac{x_1 + \Phi}{x_2}) + F(x_2^2 + x_1^2 + 2x_1\Phi) + h(x_3)$$

F and h are differentiable functions to be chosen.

Since $\frac{\partial H_a}{\partial x_3}$ only depends on x_3,

$$h(x_3) = -\omega_* \tilde{x}_3 + \frac{\alpha_2}{2} \tilde{x}_3^2$$

where $\tilde{(\cdot)} \triangleq (\cdot) - (\cdot)_*$, and $\alpha_2 > 0$. This choice yields $K_3 = -\omega_* + \alpha_2 \tilde{x}_3$.

The equilibrium assignment and Lyapunov stability conditions reduce to

$$f(\bar{z}) = -\frac{1}{2L_q} \frac{\bar{z}}{\bar{z} + \Phi^2}\ , \quad \left.\frac{\partial f(z)}{\partial z}\right|_{z=\bar{z}} > \frac{1}{4L_q} \frac{\bar{z} - \Phi^2}{(\bar{z} + \Phi^2)^2}$$

Propose $f(z) = -\frac{1}{2L_q}\frac{\bar{z}}{z+\Phi^2}$, which yields

$$\begin{aligned}
\frac{\partial H_a}{\partial x_1} &= \frac{\tau_l/P}{x_2^2+(x_1+\Phi)^2}\left[x_2 - \frac{L_q\tau_l}{P\Phi^2}(x_1+\Phi)\right] \\
\frac{\partial H_a}{\partial x_2} &= -\frac{\tau_l/P}{x_2^2+(x_1+\Phi)^2}\left[(x_1+\Phi) + \frac{L_q\tau_l}{P\Phi^2}x_2\right] \\
\frac{\partial H_a}{\partial x_3} &= -\omega_* + \alpha_2\tilde{x}_3
\end{aligned}$$

- x_* is asymptotically stable, but the initial conditions

$$\{(x_1(0)+\Phi)^2 + x_2(0)^2 \geq \epsilon > 0\}$$

and the load torque is different from zero.

- Load torque is unknown

$$\begin{aligned}
\frac{d\hat{\omega}}{dt} &= \frac{P}{J}\left(\gamma x_1 + \frac{\Phi}{L_q}\right)x_2 - l_1(\hat{\omega}-\omega) - \frac{1}{J}\hat{\tau}_l \\
\frac{d\hat{\tau}_l}{dt} &= l_2(\hat{\omega}-\omega)
\end{aligned}$$

with $\gamma \triangleq \frac{1}{L_q} - \frac{1}{L_d}$, and l_1, l_2 some positive design parameters.

C. Isotropic Interconnection Controller

- Modify the interconnection matrix to "emulate" an isotropic machine ($L_d = L_q = L$) which is easier to control.

- PCH model with $g(x)$, ζ, $\mathcal{R}$ and u as before, and

$$\mathbf{J}(x) = \begin{bmatrix} 0 & \frac{LP}{J} x_3 & 0 \\ -\frac{LP}{J} x_3 & 0 & -\Phi \\ 0 & \Phi & 0 \end{bmatrix}$$

Propose

$$\mathbf{J}_d(x) = \begin{bmatrix} 0 & L_0 x_3 & 0 \\ -L_0 x_3 & 0 & -\Phi \\ 0 & \Phi & 0 \end{bmatrix}$$

where L_0 is a parameter to be defined.

Proposition

The control law

$$\begin{bmatrix} v_d \\ v_q \end{bmatrix} = \begin{bmatrix} (\frac{L_0}{L_q} - \frac{P}{J})x_2x_3 - R_s\alpha_1 x_1 \\ -(\frac{L_0}{L_d} - \frac{P}{J} + L_0\alpha_1)x_1x_3 + \Phi(\frac{P}{J}x_{3*} - \alpha_2\tilde{x}_3) \end{bmatrix} +$$

$$+ \begin{bmatrix} -R_s & L_0x_3 \\ -L_0x_3 & -R_s \end{bmatrix} \begin{bmatrix} \frac{\gamma}{2\Phi}(x_2^2 - x_{2*}^2) \\ \frac{\gamma}{\Phi}x_1x_2 - \frac{1}{L_q}x_{2*} \end{bmatrix}$$

where L_0 is arbitrary

$$\begin{aligned} \alpha_2 + \frac{P}{J} &> 0 \\ (\alpha_1 + \frac{1}{L_d})\frac{1}{L_q} &> \frac{\gamma^2}{\Phi^2}x_{2*}^2 \end{aligned}$$

ensures x_* is GAS[a] and energy–Lyapunov function

$$H_d = \frac{1}{2}x^\top \mathbf{D}^{-1}x + \frac{\gamma}{2\Phi}x_1(x_2^2 - x_{2*}^2) - \frac{1}{L_q}x_{2*}x_2 +$$

$$\frac{\alpha_1}{2}x_1^2 - \frac{P}{J}x_{3*}\tilde{x}_3 + \frac{\alpha_2}{2}\tilde{x}_3^2$$

◇ ◇ ◇

[a] With domain of attraction the *whole* state space $\mathcal{R}^3$ minus a set of zero measure.

D. Connections with Current Practice

To recover a linear dynamics in the electrical subsystem it is common tocancel the nonlinear terms.

$$\begin{aligned} v_d &= -\omega L i_q + v_{d1} \\ v_q &= \omega L i_d + \omega \Phi + v_{q1} \end{aligned}$$

Drawback is lack of robustness. An alternative

$$\begin{aligned} v_d &= R_s i_{d*} - \omega L i_{q*} \\ v_q &= R_s i_{q*} + \omega L i_{d*} + \omega_* \Phi \end{aligned}$$

where $i_{q*} = \frac{\tau_l}{P\Phi}$. The closed loop incremental dynamics is then

$$\begin{aligned} L\frac{d\tilde{i}_d}{dt} &= -R_s \tilde{i}_d + \omega L \tilde{i}_q \\ L\frac{d\tilde{i}_q}{dt} &= -R_s \tilde{i}_q - \omega L \tilde{i}_d - \Phi \tilde{\omega} \\ \frac{J}{P}\frac{d\tilde{\omega}}{dt} &= \Phi \tilde{i}_q \end{aligned}$$

Asymptotically table if the load torque is known. In practice current reference generated by PI controller in the outer loop. Stability?

In the isotropic rotor case $\gamma = 0$, and IDA has the same form by setting $\alpha_1 = \alpha_2 = 0$ and $L_0 = \frac{PL}{J}$. Only difference that the desired values for the currents are generated by the nonlinear observer.

Underactuated Kirchhoff's equations

A. The model

- Ellipsoidal rigid body submerged in an ideal fluid and assume that the center of gravity of the body coincides with the center of buoyancy.
- **Kirchhoff equations** *(Leonard, Automatica'97)*

$$
\begin{aligned}
\dot{\Pi}_1 &= \left(\tfrac{1}{J_3} - \tfrac{1}{J_2}\right)\Pi_2\Pi_3 + \left(\tfrac{1}{M_3} - \tfrac{1}{M_2}\right)P_2P_3 + T_1 \\
\dot{\Pi}_2 &= \left(\tfrac{1}{J_1} - \tfrac{1}{J_3}\right)\Pi_3\Pi_1 + \left(\tfrac{1}{M_1} - \tfrac{1}{M_3}\right)P_3P_1 + T_2 \\
\dot{\Pi}_3 &= \left(\tfrac{1}{J_2} - \tfrac{1}{J_1}\right)\Pi_1\Pi_2 + \left(\tfrac{1}{M_2} - \tfrac{1}{M_1}\right)P_1P_2 \\
\dot{P}_1 &= \tfrac{P_2\Pi_3}{J_3} - \tfrac{P_3\Pi_2}{J_2} \\
\dot{P}_2 &= \tfrac{P_3\Pi_1}{J_1} - \tfrac{P_1\Pi_3}{J_3} \\
\dot{P}_3 &= \tfrac{P_1\Pi_2}{J_2} - \tfrac{P_2\Pi_1}{J_1} + F_3
\end{aligned}
$$

where Π_1, Π_2 and Π_3 (P_1, P_2 and P_3) angular (linear) momentum, $J_1 > 0$, $J_2 > 0$ and $J_3 > 0$ principal moments of inertia and $M_1 > 0$, $M_2 > 0$ and $M_3 > 0$ terms of the inertia matrix,

- T_1, T_2 and F_3 manipulated variables $\Rightarrow$ the system is *locally strongly accessible.*

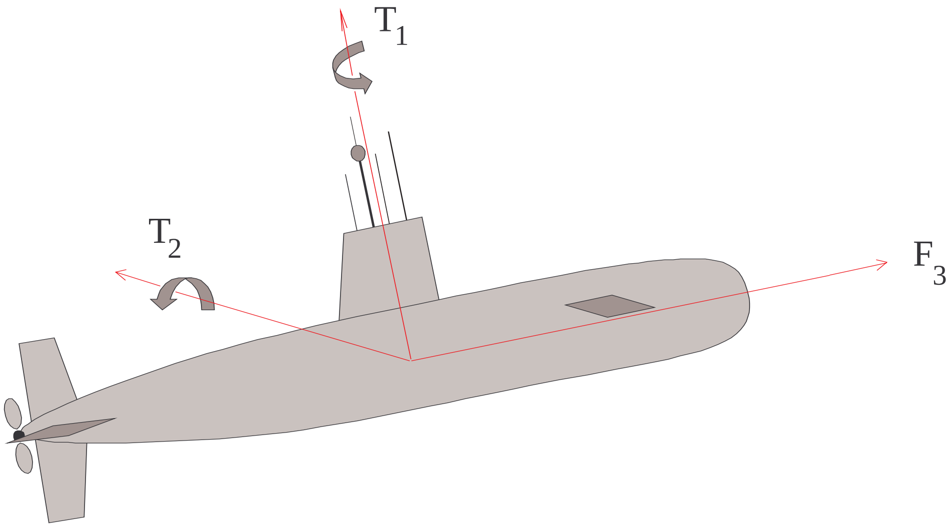

Figure 28: Underwater vehicle with actuator configuration

- Stabilization of equilibria

$$x_e = \begin{cases} \mathrm{col}(0,0,0,0,\bar{P}_2,0), & \text{forward / reverse} \\ \mathrm{col}(0,0,0,\bar{P}_1,0,\bar{P}_3), & \text{diving / rising with f / r} \end{cases}$$

- Port-controlled Hamiltonian (PCH) description

$$\dot{x} = (J(x) - R(x)) \left(\frac{\partial H}{\partial x} \right)^T + Gu.$$

– $x = \mathrm{col}(\Pi_1, \Pi_2, \Pi_3, P_1, P_2, P_3) \in \mathbb{R}^6$,

– $u = \mathrm{col}(T_1, T_2, F_3) \in \mathbb{R}^3$,

– $H(x) \geq c$ energy function,

– $J(x) = -J^\top(x)$ interconnection,

– $R(x) = R^\top(x) \geq 0$ dissipation.

B. Selective damping

Proposition

Solutions of PDE

$$G^{\perp} J(x) \left(\frac{\partial H_a}{\partial x} \right)^T = 0,$$

ensure

$$u(x) = [G^\top G]^{-1} G^\top \left[(J(x) - R_a(x)) \left(\frac{\partial H_a}{\partial x} \right)^\top - R_a(x) \left(\frac{\partial H}{\partial x} \right)^\top \right]$$

stabilizes x_e, for all $R_a(x) = R_a^\top(x) \geq 0$ and

$$\boxed{\mathrm{Im}(R_a(x)) \subseteq \mathrm{Im}(G).}$$

B. Main results

Key Lemma

All solutions of the PDE are of the form

$$\prod_{i \in \mathcal{I}} \phi_i(\psi_j),$$

with $\mathcal{I}$ a finite set of indexes, $j = 1, 2, 3$, differentiable $\phi_i(\cdot)$ and

$$\begin{aligned} \psi_1 &= P_3 \\ \psi_2 &= P_1^2 + P_2^2 \\ \psi_3 &= \Pi_1 P_1 + \Pi_2 P_2 + \Pi_3 P_3 \end{aligned}$$

Proposition

– The equilibrium is almost[a] **globally asymptotically** stable.

– Suppose $M_1 > M_2$. Then, the steady rising / diving with forward / reverse motion equilibrium is **locally asymptotically** stable.

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[a]For all initial conditions, except a set of measure zero, trajectories converge asymptotically to x_e.

C. Simulation results

Parameters from *(Leonard, Automatica'97)*

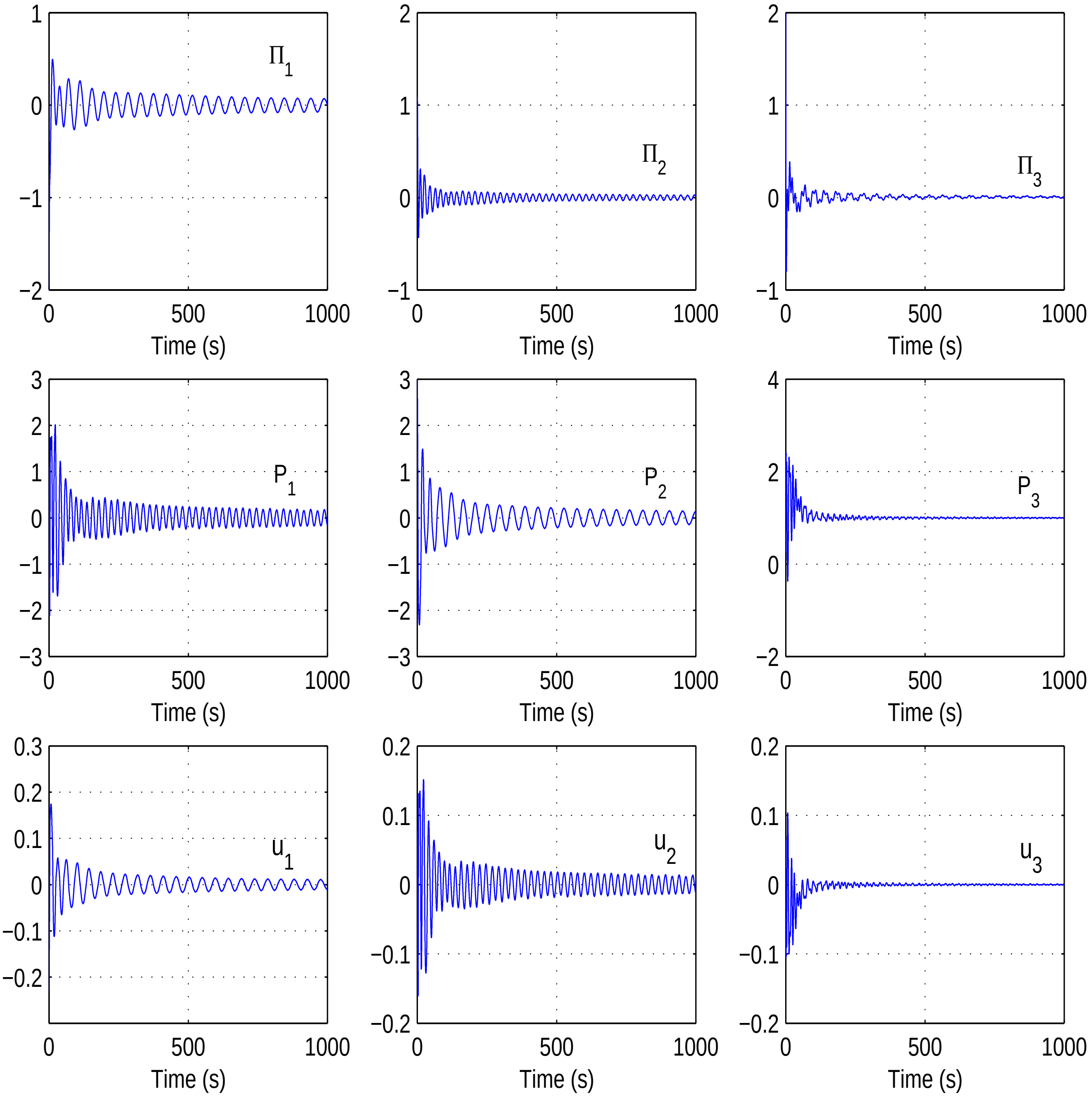

Figure 29: Steady forward / reverse motion with $\bar{P}_3 = 1$.

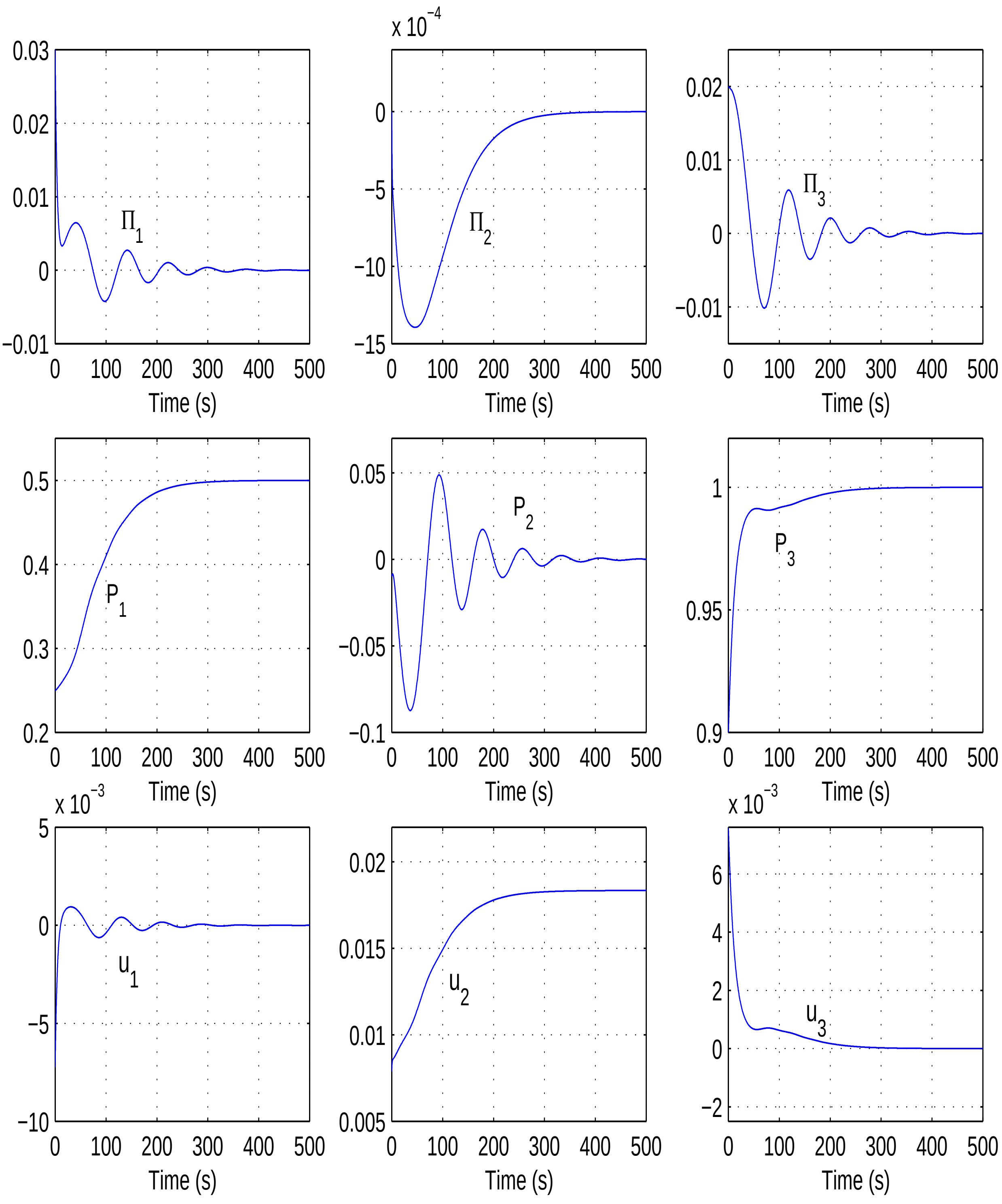

Figure 30: Steady rising / diving with forward / reverse motion with $\bar{P}_1 = 1/2$ and $\bar{P}_3 = 1$.

8. Current and future research

♡ Electrical motors: Synchronous and stepping.

♡ Full characterization of underactuated mechanical systems.

♡ "Controlled Lagrangians" *(Bloch et al.,'98)* $\subset$ IDA.

♡ **Non–switching** stabilizing solutions For pendulum with the chariot, ball and beam and gyroscopic pendulum.

• Dualize the problem, fixing H_d to some desired form, and trying to solve the algebraic equation for J_d, *(Fujimoto/Sugie, '2000)*.

• *Gyroscopic forces* play a fundamental role in electromechanical applications, where the desired equilibrium does not occur at zero kinetic energy.

Open problems

• Solvability of the PDE with "boundary conditions".

• Trajectory tracking

• Adaptation: On–line identification of the parameters of the energy function using energy–balancing.

• Dynamic extension.