A Geometric Method for Computation of Datum Reference Frames

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***Abstract*—A datum reference frame (DRF) is a coordinate system used to locate and orient part features. Constructing a DRF from a set of datum features is a complicated process involving: a) specifying a valid combination and the precedence of the datum features which define the DRF; b) developing datum from datum features of the part; and c) determing the position and orientation of the DRF from the datums. In this paper, we develop a geometric theory for establishing DRFs. The theory is based on the observation that a datum feature such as a plane, a cylinder or a sphere has a symmetry subgroup G_0 under the action by the group $\mathrm{SE}(3)$ of rigid motions in $\mathbb{R}^3$. Thus, the configuration space of a datum feature can be identified with the homogeneous space $\mathrm{SE}(3)/G_0$, and the problem of datum development can be posed as a minimization problem in $\mathrm{SE}(3)/G_0$. We give conditions under which a datum feature qualifies to be a secondary or a tertiary datum. We present a sequential procedure that transforms the primary, secondary and tertiary datum problems as a minimization or a constrained minimization problem in the homogeneous spaces of $\mathrm{SE}(3)$. We develop simple algorithms to solve these problems, and give simulation results illustrating efficiency and simplicity of the approach.**

***Index Terms*—Datum reference frame, Euclidean groups, homogeneous spaces, Lie groups.**

I. INTRODUCTION

A DATUM reference frame (DRF) is a coordinate system used to locate and orient part features. To verify location tolerances of a part, it is necessary that a DRF be first established using measurement data from relevant datum features. Establishing a DRF from datum features is a complicated process involving: a) specifying a valid combination and the procedence of the datum features which define the DRF; b) developing datums from datum features of the part; and c) determining the position and orientation of the DRF from the datums. Traditionally, a DRF is established using hard-gauges which simulate perfect-form DRFs. This process is apparently too costly, inflexible and unsuited for automatic inspection of a large quantity of parts. Advances in computer numerical control (CNC) technology makes coordinate measuring machines (CMMs) a widely accepted tool in tolerance inspection. Unlike the hard-gauge approach, a CMM-based approach produces simulated gauges by applying data analysis algorithms to measurement data obtained from the actual datum features. This overcomes all shortcomings associated with the hard-gauge approach and is applicable for automated inspection of a large quantity of parts. To fully realize the advantages derived from CMMs it is, however, necessary that we have precise mathematical definitions of datum features and DRFs so that their geometric meanings can unambigously interpreted. Furthermore, accurate and efficient algorithms are needed to determine datums and DRFs from measurement data.

ANSI Y14.5M defines the set of allowable datum features including planes, pairs of parallel planes (width features), cylinders and spheres. It specifies DRFs on a case by case basis. Thus, the method of establishing a DRF from datums depends on the type of datum features, the precedence of datum features (primary, secondary, and tertiary), and the material conditions (maximum material conditions [MMC], least material conditions [LMC], regardless of feature size [RFS]). All existing approaches use a distinct formulation for each valid combination of datum features and develop evaluation algorithms accordingly [3], [2], [15], [4], [10], [13], [8].

The new ANSI Y14.5.1M [8] standard, or the so-called the mathematical companion of ANSI Y14.5M, attempts to provide precise definitions for all datum features. It defines rules for constructing DRFs from datum sets. It lists all fifty-two possible combinations under which a set of datum features can define a DRF. To establish a datum from an actual datum feature, a set of rules has to be followed. This makes the approach unsuitable for CMM implementation. Furthermore, it is cumbersome to maintain all the conditions for validity of DRFs.

Estesami [2] studied the establishment of DRFs in the two-dimensional case. In his study, DRFs were viewed as “datum-priority-frames” and all nine canonical coordinate systems were enumerated based on combinations of lines, parallel lines and points in $\mathbb{R}^2$. The problem of establishing DRFs was formulated as a constrained optimization problem and the Lagrange multiplier technique was used to solve the problem. This approach conforms with the ANSI standard but is tedious and inefficient. Furthermore, it is difficult to extend the approach to three dimensional case as the number of possible datum combinations increases dramatically.

Turner and his coworkers [15], [13], [14] studied extensively problems arising from establishment of DRFs. In their study, DRFs were represented by a chain of datum features. The validity of a DRF is verified using a symbolic approach known as CRR which consists of nine computation rules. Since their rep-

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Manuscript received September 10, 1999; revised June 26, 2000. This paper was recommended for publication by Associate Editor P. Banerjee and Editor N. Viswanadham upon evaluation of the reviewers’ comments. This work was supported in part the RGC under Grant HKUST 685/95E, Grant HKUST 555/94E, Grant HKUST 193/93E, Grant HKUST 6220/98E, and Grant HKUST6221/99E, and HKUST Research Infrastructure Grant R1/92/93.EG18, Grant CRC98/01.EG02, and in part by the Department of Electrical and Electronic Engineering of the Hong Kong University of Science and Technology.

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Publisher Item Identifier S 1042-296X(00)11651-9.

resentation of DRFs is not mathematically precise, the computation of a coordinate system is not efficient. Furthermore, only fully constrained DRFs can be determined using their approach.

Despite much of previous research effort in DRFs, we still lack a systematic and implementable method for establishing DRFs from actual datum features. In this paper, we propose a geometric approach for formulation and establishment of DRFs. Our method is based on an important observation that a datum feature such as a plane, a cylinder or a sphere has a symmetry subgroup G_0 under the action of the Euclidean group $\mathrm{SE}(3)$. Thus, the configuration space of a datum feature can be identified with the homogeneous space $\mathrm{SE}(3)/G_0$, and the problem of datum development can be formulated as a minimzation problem in $\mathrm{SE}(3)/G_0$. We give conditions under which a datum feature qualifies to be a secondary or tertiary datum. We present a sequential procedure that transforms the primary, secondary and tertiary datum problems as a sequence of (constrained) minimizaion problems in the homogeneous spaces of $\mathrm{SE}(3)$. We develop simple algorithms to solve these problems and thus the problem of establishing DRFs from actual datum features. We present simulation results showing simplicity, efficieny and correctness of the approach.

The paper is organized as follows. In Section II, we review several useful concepts of the Euclidean group $\mathrm{SE}(3)$ and its homogeneous spaces. In Section III, we show that datum development from actual features can be formulated as a (constrained) minimization problem in $\mathrm{SE}(3)/G_0$, where $G_0 \subset \mathrm{SE}(3)$ is the symmetry subgroup of the feature. In Section IV, we give a geometric algorithm for establishing coordinate systems from datums. In Section V, we present simulation results illustrating simplicity, efficiency and correctness of our method.

II. CONFIGURATION SPACES OF SYMMETRIC FEATURES

In this section, we briefly review properties of the Euclidean group $\mathrm{SE}(3)$ and its homogeneous spaces. Detailed treatment on these topics can be found in [1] and [7].

In the robotics literature [11], the configuration space of a rigid body moving in $\mathbb{R}^3$ is identified with the Euclidean group. More specifically, by attaching a coordinate frame to the object, a configuration of the object is given by $g = (p, R)$, where $p \in \mathbb{R}^3$ is the position and $R \in \mathrm{SO}(3)$ the orientation matrix of the object frame relative to some reference frame. In homogeneous coordinates, g is written as

$$g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

and the set of such matrices is denoted $\mathrm{SE}(3)$. Using matrix multiplication as the group operation, $\mathrm{SE}(3)$ is seen to form a group, known as the special Euclidean group of $\mathbb{R}^3$.

Denote by $\mathrm{so}(3)$ the set of 3×3 skew-symmetric matrices and identify it with $\mathbb{R}^3$ via the isomorphism

$$\hat{}: \mathbb{R}^3 \to so(3) : \omega \mapsto \hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.$$

The set of 4×4 matrices of the form

$$\hat{\xi} := \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}, \qquad \omega, v \in \mathbb{R}^3$$

denoted $\mathrm{se}(3)$, forms a linear space known as the Lie algebra of $\mathrm{SE}(3)$. For each $\hat{\xi} \in \mathrm{se}(3)$, the vector $\xi = (v, \omega) \in \mathbb{R}^6$ is called the twist coordinates of $\hat{\xi}$. Let $\xi_i, i = 1, \ldots, 6$, be the canonical basis of $\mathbb{R}^6$, then $\{\hat{\xi}_i\}_{i=1}^6$ gives a basis of $\mathrm{se}(3)$. The exponential map of SE(3)

$$\exp : \mathrm{se}(3) \to \mathrm{SE}(3) : \hat{\xi} \mapsto e^{\hat{\xi}}$$

defines the canonical coordinates of $\mathrm{SE}(3)$ near the identity [1], [11].

Consider now rigid motions of a symmetric feature such as an (infinite) plane. Because of the symmetry, the configuration space of the plane can not be uniquely represented by elements of $\mathrm{SE}(3)$. In other words, the set of transformations differing from each other by a rotation about the normal of the plane and a translation along the plane would represent the same plane. For instance, let the reference plane be the xy-plane, then it has the symmetry subgroup

$$G_0 = \{e^{m_1\hat{\xi}_1 + m_2\hat{\xi}_2 + m_3\hat{\xi}_6} \,|\, m_1, m_2, m_3 \in \mathbb{R}\}$$

where $\hat{\xi}_1$ and $\hat{\xi}_2$ denote infinitesimal translations on the xy-plane and $\hat{\xi}_3$ infinitesimal rotation about the z-axis. Two configurations $g_1, g_2 \in \mathrm{SE}(3)$ are seen to represent the same plane, or $g_1 \sim g_2$, if and only if $g_1 \cdot g_2^{-1} \in G_0$. Thus, by identifying equivalent Euclidean transformations, the configuration space of the plane is uniquely represented by the quotient space

$$\mathrm{SE}(3)/G_0 = \{gG_0 \,|\, g \in \mathrm{SE}(3)\}.$$

An element $gG_0 \in \mathrm{SE}(3)/G_0$, also denoted $[g]$ or simply $g \in \mathrm{SE}(3)/G_0$, represents an equivalent class of Euclidean transformations. Given $g = (R, p) \in \mathrm{SE}(3)/G_0$, where $R = [v_1, v_2, v_3] \in \mathrm{SO}(3)$, the corresponding plane is located at $p \in \mathbb{R}^3$ and with normal direction $v_2 \in \mathbb{R}^3$. Conversely, given a plane $P_{(v,q)}$ that passes through $q \in \mathbb{R}^3$ and has normal direction $v \in \mathbb{R}^3$, the set of Euclidean transformations taking the reference plane to $P_{(v,q)}$ is given by

$$\begin{bmatrix} [v_1, v_2, v] & q \\ 0 & 1 \end{bmatrix} \in \mathrm{SE}(3)$$

where $v_1, v_2 \in \mathbb{R}^3$ are unit vectors such that $[v_1, v_2, v] \in \mathrm{SO}(3)$.

Example 1 Rigid Motions of a Cylinder: Consider rigid motions of a cylinder in $\mathbb{R}^3$, that is, motions of its axis. Let the reference configuration be the z-axis. Then the symmetry subgroup consists of rotations about and translations along the axis, i.e.,

$$G_0 = \{e^{m\hat{\xi}_3 + m_2\hat{\xi}_6} \,|\, m_1, m_2 \in \mathbb{R}\}.$$

The configuration space of a cylinder moving in $\mathbb{R}^3$ is then given by $\mathrm{SE}(3)/G_0$.

Example 2 Rigid Motions of a Sphere: Consider rigid motions of a sphere moving in $\mathbb{R}^3$. The symmetry subgroup of the nominal sphere with center at the origin consists of all rotations about the origin

$$G_0 = \{e^{m_1\hat{\xi}_4 + m_2\hat{\xi}_5 + m_3\hat{\xi}_6} \mid m_1, m_2, m_3 \in \mathbb{R}\}.$$

The configuration space of the sphere is given by $\mathrm{SE}(3)/G_0$, which can be identified with $\mathbb{R}^3$, the coordinates of its center.

Given a subgroup $G_0 \subset \mathrm{SE}(3)$, the set $\mathrm{SE}(3)/G_0$ is a differentiable manifold of dimension $(6 - \dim(G_0))$. It is called a *homogeneous space* of $\mathrm{SE}(3)$ [1] because it admits a transitive action of $\mathrm{SE}(3)$ given by

$$\mu : \mathrm{SE}(3) \times \mathrm{SE}(3)/G_0 \to \mathrm{SE}(3)/G_0 : (h, gG_0) \mapsto hgG_0. \tag{1}$$

In the case of a plane, (1) indicates that any two planes are related by a Euclidean transformation. Physically, hgG_0 is a plane obtained by applying the Euclidean transformation $h \in \mathrm{SE}(3)$ to the plane gG_0, while gG_0 itself is interpreted as the plane obtained by applying $g \in \mathrm{SE}(3)$ to the reference plane.

Example 3 Exponential Coordinates of $\mathrm{SE}(3)/G_0$*:* For the computation of DRFs, a useful coordinate system on $\mathrm{SE}(3)/G_0$, called the exponential coordinates can be defined as follows. First, denote by $\mathcal{G}_0$ the Lie algebra of G_0, and choose a complementary space $\mathcal{M}_0$ such that $\mathrm{se}(3)$ is given by direct sum of $\mathcal{M}_0$ and $\mathcal{G}_0$

$$\mathcal{M}_0 \oplus \mathcal{G}_0 = \mathrm{se}(3).$$

It is not difficult to see that the map

$$\psi : \mathcal{M}_0 \oplus \mathcal{G}_0 \to \mathrm{SE}(3) : (\hat{m}, \hat{h}) \mapsto \exp(\hat{m})\exp(\hat{h})$$

is a local diffeomorphism.

Finally, let $g \in \mathrm{SE}(3)$ be a representative element of gG_0, and decompose g into

$$g = \exp(\hat{m})\exp(\hat{h})$$

where $\hat{m} \in \mathcal{M}_0$ and $\hat{h} \in \mathcal{G}_0$. Let $r = \dim(\mathrm{SE}(3)/G_0)$ and let $(\hat{\eta}_1, \ldots, \hat{\eta}_r)$ be a basis of $\mathcal{M}_0$. Express $\hat{m}$ as

$$\hat{m} = \sum_{i=1}^{r} m_i \hat{\eta}_i.$$

Then, the map

$$\tilde{\psi} : \mathrm{SE}(3)/G_0 \to \mathbb{R}^r : gG_0 \mapsto (m_1, \ldots, m_r)$$

is well defined, and gives the desired coordinate system for $\mathrm{SE}(3)/G_0$.

Finally, consider rigid motions of a composite feature. A composite feature, often used in defining a coordinate system, consists of a set of features in which each component feature is constrained with regard to its position and relative orientation. A simple example of a composite feature is a half cylinder which can be thought of as an (infinite) cylinder (feature A) intersected by a plane (feature B) that is perpendicular to it. Let G_A and G_B be, respectively, the symmetry subgroup of the component features A and B as in Example 1 and 2, then the symmetry subgroup G_{AB} of the composite feature is given by

$$G_{AB} = G_A \cap G_B = \{e^{m\hat{\xi}_6} \mid m \in \mathbb{R}\}$$

and its configuration space by $\mathrm{SE}(3)/G_{AB}$.

As another example, consider Fig. 4(a) where the planar feature A, the cylindrical feature B and the width feature C are combined to define a DRF which is then used to locate and orient the hole. Here, feature B is constrained to be perpendicular to A and C is constrained to be perpendicular to A while its center plane passes through the axis of B. Without loss of generality, we assume that a reference frame is defined as shown in the figure. The composite feature A-B-C defines a coordinate system relative to which the hole feature can be referenced. In terms of the ANSI Y14.5M specification for DRFs, the degrees of freedom of rigid motions are sequentially constrained by the primary datum (feature A), the secondary datum (feature B) and the tertiary datum (feature C). The primary datum, because of its symmetry, can only determine degrees of freedom in $\mathrm{SE}(3)/G_A$, while leaving transformations in G_A undetermined. Here, G_A is the symmetry group of the primary datum given by

$$G_A = \{e^{m_1\hat{\xi}_1 + m_2\hat{\xi}_2 + m_3\hat{\xi}_6} \mid m_1, m_2, m_3 \in \mathbb{R}\}$$

with

$$\mathcal{G}_A = \mathrm{span}\{\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3\}.$$

The secondary datum B is used to eliminate the free transformations left by A. In other words, the free transformations left using the combination of A-B are given by

$$G_{AB} = G_A \cap G_B = \{e^{m_1\hat{\xi}_6} \mid m_1 \in \mathbb{R}\}$$

i.e., the set of rotations about the z-axis. We see that addition of B does not completely constrain the rigid motions left by A. With the addition of the tertiary datum C, we have the the symmetry subgroup of the composite feature A-B-C given by

$$G_{ABC} = G_A \cap G_B \cap G_C = I.$$

Thus, the control frame A-B-C completely constrains rigid motions of the workpiece and defines a valid coordinate system which can be used to locate and orient the hole feature.

More generally, let a composite feature F be composed of a set of symmetric features $\{S^1, \ldots, S^m\}$, each with a symmetry subgroup S_0^i, then the symmetry subgroup of F is given by

$$F_0 = \bigcap_{i=1}^{m} S_0^i$$

and its configuration space by $\mathrm{SE}(3)/F_0$.

III. FORMULATION OF DRFs

According to the ANSI Y14.5M standard, establishment of a DRF depends on the type of datum features, the precedence of datum features, and the material conditions of datum features of size. In the standard, a primary datum is a perfect-form feature which “contacts” the corresponding feature, and establishes a partially determined DRF while leaving free transformations in its symmetry subgroup undetermined. A secondary datum is a perfect-form feature which is subject to spatial constraints relative to the primary datum, and "contacts" the corresponding datum feature. The callout of a secondary datum is used to eliminate or further constrain the undetermined transformations left by the primary datum. A a tertiary datum is a perfect-form feature which is subject to spatial constraints relative to the primary and the secondary datums, and "contact" the corresponding datum feature. The callout of a tertiary datum is used to eliminate the undetermined transformations left by the higher-precedence datums. The material condition of a datum feature of size may establish a set of candidate datums from the feature. In this case, the established DRF is the one which minimizes deviations of toleranced features from the nominal features.

In this section, we will use the theory developed in the previous section to formulate DRFs constructed by combination of planes, width features, cylinders and spheres. For notational convenience, we denote by A the primary datum feature, B the secondary datum feature, and C the tertiary datum feature.

A. Primary Datum

The primary datum is used to determine a DRF such that the position and orientation of the datum feature match closely to that of the nominal feature. Thus, the configuration space of the established DRF is just that of the primary datum feature, i.e., $\mathrm{SE}(3)/G_A$, where G_A is the symmetry subgroup of the feature. According to the specifications of the standard Y14.5.1M [8] and the minimum zone principle, we give the definition of a primary datum constructed from a planar feature, a width feature, a cylindrical feature and a spherical feature.

Definition 1 (Planar Datum): The datum constructed from a planar datum feature is a supporting plane of the feature satisfying the following two conditions.

1) All points on the datum feature lie on the material side of the datum plane.
2) The maximum deviation of points on the datum feature from the datum plane is minimized.

Let $Y = \{y_i\}_{i=1}^n, y_i \in \mathbb{R}^3$, be a set of measurement points from the datum feature. From the above definition, the corresponding datum plane P can be obtained by solving the mini-max problem

$$\min_{g \in \mathrm{SE}(3)/G_A} \max_{y_i \in Y} d(g, y_i) \tag{2}$$

where

$$d(g, y_i) = \langle g^{-1} y_i - x_i, v \rangle$$

is the directional distance function from y_i to $P, g^{-1}y_i$ the transformed point of $y_i, x_i \in P$ the corresponding point of y_i, and v the normal vector of the nominal datum plane directing to the material side.

Note that the objective function (2) is nondifferentiable. A standard technique in optimization [12] shows that by extending the configuration space to $\mathrm{SE}(3)/G_A \times \mathbb{R}$, problem (2) can be transformed to a minimization problem with a linear objective function and nonlinear constraints. The objective function in the extended space is

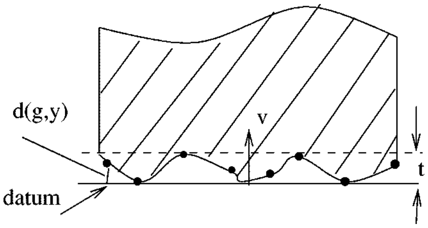


Fig. 1. Planar datum feature.

$$f(g, t) = t \tag{3}$$

and the constraints are

$$0 \leq d(g, y_i) \leq t, \qquad i = 1, \ldots, n. \tag{4}$$

Inequality (4) guarantees that all measurement points lie within the tolerance zone, as shown in Fig. 1. The optimal solution $g_a \in \mathrm{SE}(3)/G_A$ of the problem gives the DRF established by A. The optimal value of $f(g,t)$ gives the maximum deviation of A from the datum plane, that is

$$t = \max_{y_i \in Y} d(g, t).$$

Definition 2 (Datum of Width Feature): The datum established by a width feature is the center plane of the mating width feature. Because a width feature is a feature of size, the position and orientation of the center plane depend on the material condition of the datum callout.

1) If the datum is called out in RFS condition, the datum plane is the center plane of the actual mating feature.
2) If the datum is called out in MMC/LMC condition, then the datum plane is the set of center planes of all the virtual features in MMC/LMC. Each center plane is referred to as a candidate datum. The datum plane should be chosen so that the tolerance values of the referenced features are minimized.

Given data sets $Y_1 = \{y_{1i} \in \mathbb{R}^3\}_{i=1}^n$ measured from one component plane, and $Y_2 = \{y_{2j} \in \mathbb{R}^3\}_{j=1}^m$ the other component plane, the datum plane of an external feature in RFS can be obtained by minimizing the size of the mating feature, given by

$$\min_{g \in \mathrm{SE}(3)/G_A} \max_{y_i \in Y_1 \cup Y_2} |d(g, y_i)|$$

where $d(g, y_i)$ is the directional distance from point y_i to the datum plane

$$d(g, y_i) = \langle g^{-1} y_i - x_i, v \rangle$$

and $v \in \mathbb{R}^3$ is the normal vector of the width feature. Extending the configuration space to $\mathrm{SE}(3)/G_A \times \mathbb{R}$ leads to a transformed minimization problem with the objective function

$$f(g, t) := t$$

and the constraints

$$\begin{cases} d(g, y_i) \leq t, & i = 1, \ldots, n \\ d(g, y_j) \geq -t, & j = 1, \ldots, m \end{cases} \tag{5}$$

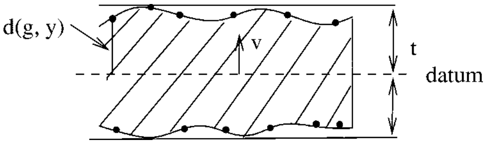


Fig. 2. External datum width feature.

Inequality (5) constrains all measurement points to lie within the mating feature, as shown in Fig. 2. Note that the optimal value of the objective function is the half size of the width feature

$$t = \max_{y_i \in Y_1 \cup Y_2} d(g, y_i).$$

Reversing (5) and maximizing the objective function yields the datum formulation problem for an internal width feature.

If the datum feature is called out in MMC, the datum plane of an external width feature is a set of planes satisfying the following inequalities:

$$\begin{cases} d(g, y_i) \leq \frac{1}{2} t_{\mathrm{mmc}}, & i = 1, \ldots, n \\ d(g, y_j) \geq -\frac{1}{2} t_{\mathrm{mmc}}, & j = 1, \ldots, m \end{cases} \tag{6}$$

where t_{mmc} is the size of the feature in MMC.

Reversing Inequality (6) leads to the formulation of a set of candidate planar datums for an internal feature.

Definition 3 (Datum of Cylindrical Feature): The datum established from a cylindrical feature is the axis of the mating feature. Because it is a feature of size, the location and orientation of the axis datum depends on the material condition of the datum callout.

1) If the datum is called out in RFS condition, the datum axis is that of the actual mating feature.
2) If the datum is called out in MMC/LMC condition, then the datum axis is a set of axes of all the virtual features in MMC/LMC. Each axis is referred to as a candidate datum. The datum axis should be chosen so that the tolerance values of the referenced features are minimized.

Given a set of measurement point $Y = \{y_i\}_{i=1}^{n}$, the datum axis of an external feature in RFS can be obtained by minimizing the size of the mating feature, i.e.,

$$\min_{g \in \mathrm{SE}(3)/G_A} \max_{y_i \in Y} d(g, y_i)$$

where $d(g, y_i)$ is the distance from the transformed point $g^{-1} y_i$ to the datum axis

$$d(g, y_i) = \langle g^{-1} y_i - x_i, n_i \rangle$$

$x_i \in \mathbb{R}^3$ the point on the axis nearest to the transformed point $g^{-1} y_i$, and $n_i \in \mathbb{R}^3$ the unit vector in the direction of $(g^{-1} y_i - x_i)$, as shown in Fig. 3. Similarly, by extending the configuration space to $\mathrm{SE}(3)/G_A \times \mathbb{R}$ we obtain the transformed minimization problem, with

$$f(g, t) = t \tag{7}$$

and the constraints

$$d(g, y_i) \geq t, \qquad i = 1, \ldots, n. \tag{8}$$

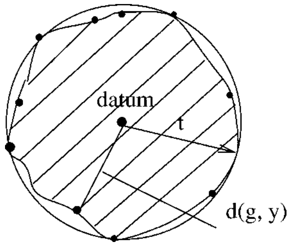


Fig. 3. External datum cylindrical feature.

Inequality (8) constrain all the measurement points to lie within the actual mating feature. The optimal g gives the cylindrical datum. The optimal value of the objective function t is the radius value of the actual mating feature

$$t = \max_{y_i \in Y} d(g, y_i).$$

Reversing Inequality (8) and maximizing the objective function (7) yields the formulation of the datum axis for an internal feature.

If a cylindrical datum is called out in MMC, the datum axis of an external feature is a set of candidate axes satisfying the following inequalities:

$$d(g, y_i) \leq \frac{1}{2} t_{\mathrm{mmc}}, \qquad i = 1, \ldots, n. \tag{9}$$

Reversing Inequality (9) leads to the formulation of the set of candidate datum axes for an internal cylindrical feature.

Definition 4 (Datum of Spherical Feature): The datum established by a spherical feature is the center point of the mating feature. Because a spherical feature is a feature of size, the location of the datum point depends on the material condition of the datum callout.

1) If the datum is called out in RFS condition, the datum point is the center of the actual mating feature.
2) If the datum is called out in MMC/LMC condition, then the datum point is a set of centers of all the virtual features in MMC/LMC. Each center is referred to as a candidate datum. A datum point should be chosen so that the tolerance values of the referenced features are minimized.

Given a set of measurement points $Y = \{y_i\}_{i=1}^{n}$, $y_i \in \mathbb{R}^3$, the datum point of an external feature in RFS can be obtained by minimizing the size of the mating feature

$$\min_{g \in \mathrm{SE}(3)/G_A} \max_{y_i \in Y} d(g, y_i) \tag{10}$$

where $d(g, y_i)$ is the distance function from the transformed point $g^{-1} y_i$ to the datum point. Extending the configuration space to $\mathrm{SE}(3)/G_A \times \mathbb{R}$ yields the transformed minimization problem of (10), with

$$f(g, t) = t \tag{11}$$

and the constraints

$$d(g, y_i) \geq t, \qquad i = 1, \ldots, n \tag{12}$$

Inequality (12) constrains all measurement points to lie within the mating feature. The optimal g represents the datum point.

The optimal value of $f(g,t)$ is the radius of the actual mating sphere,

$$t = \max_{y_i \in Y} d(g, y_i).$$

Reversing Inequality (12) and maximizing the objective function (11) yields the formulation of the datum of an internal feature.

If a spherical feature is called out in MMC, the corresponding candidate point datums of an external feature satisfy the following inequalities

$$d(g, y_i) \le \frac{1}{2} t_{\text{mmc}}, \qquad i = 1, \ldots, n. \tag{13}$$

Reversing the inequalities in (13) yields the formulation of the set of candidate datum points of an internal feature in MMC.

B. Secondary Datum

A primary feature defines a partial coordinate system, that is, the set of Euclidean transformations of the form $Q_A = \{g_a G_A \,|\, g_a \in \text{SE}(3)\}$, where G_A is symmetry subgroup of the primary feature A. If a secondary datum feature B is called out, it will constrain additional degrees of freedom within G_A. Furthermore, if the symmetry subgroup of the composite feature A-B is a proper subset of G_A, i.e.,

$$G_{AB} := G_A \cap G_B \subsetneq G_A \tag{14}$$

then, the datum set A-B is said to be valid. Note that the configuration space of the composite feature A-B is given by $\text{SE}(3)/G_{AB}$.

To construct the secondary datum, let $\mathcal{M}_A$ be a complementary space of $\mathcal{G}_A$, i.e.,

$$\text{se}(3) = \mathcal{M}_A \oplus \mathcal{G}_A$$

and define $\mathcal{M}_{\mathcal{B}} \subset \mathcal{G}_A$ such that

$$\mathcal{G}_A = \mathcal{M}_B \oplus \mathcal{G}_{AB}.$$

In a neighborhood of the identity, the map

$$\mu : \text{se}(3) \to \text{SE}(3) : (\hat{m}_A, \hat{m}_B, \hat{h}_{AB}) \mapsto e^{\hat{m}_A} e^{\hat{m}_B} e^{\hat{h}_{AB}}$$

defines a local diffeomorphism, where $\hat{m}_A \in \mathcal{M}_A, \hat{m}_B \in \mathcal{M}_B$ and $\hat{h}_{AB} \in \mathcal{G}_{AB}$. Thus, an element $[g] \in \text{SE}(3)/G_{AB}$ can be locally represented by

$$g = e^{\hat{m}_A} e^{\hat{m}_B} e^{\hat{h}_{AB}}.$$

Since $\hat{m}_A$ has previously been determined from the primary datum feature A, the secondary datum is therefor computed from a similar minimization problem with Q_A replaced by

$$Q_B = \{g_a e^{\hat{m}_B} G_{AB}\}$$

where $g_a := e^{\hat{m}_A}$. Note that the addition of the secondary datum B does not change the configuration space of the primary datum A.

C. Tertiary Datum

A tertiary datum feature C is used to fully constrain the free transformations left in G_{AB}. It is said to be valid if

$$G_{ABC} = G_A \cap G_B \cap G_C = I$$

The set of transformations determined by the tertiary datum feature is given by the solution of a corresponding minimization problem with

$$Q_C = \{g_{ab} e^{\hat{m}_c} \,|\, \hat{m}_c \in G_{AB}\}$$

where g_{ab} is the transformation determined by A-B.

D. Datum Callout at MMC/LMC

When a set of datum features is called out without MMC/LMC (Maximum/Least Material Conditions) the the DRF is determined directly by the datum features, with no relation to the referenced features. However, if more than one datum feature is called out in MMC/LMC, a set of candidate coordinate systems will be established. The final DRF, according to the definition, is the one relative to which the referenced tolerance values are minimized. An extension of the previous method is now used to compute the datums called out at MMC/LMC.

Without loss of generality, we assume that B is the highest precedence datum feature called out at MMC/LMC. Thus, datum A has already determined the set of coordinate transformations of the form $g_a G_A$. The datum callouts B-C is used to eliminate the degrees of freedom in G_A. The configuration space of A-B-C is represented as

$$Q = \{g_a e^{\hat{m}_a} \,|\, \hat{m}_a \in \mathcal{G}_A\}.$$

According to the definition of datum called out at MMC/LMC, the DRF can be obtained by minimizing the tolerance function

$$f(g, F)$$

subject to the constraints

$$\begin{cases} \psi_1(g, F) \le 0 \\ \psi_2(g, B) \le 0 \\ \psi_3(g, C) \le 0 \end{cases}$$

where $g \in Q$ are the constraints that assure that all points on the toleranced feature F lie in the tolerance zone, $\psi_1(\cdot)$ the constraints that assure points in F lie in the tolerance zone, $\psi_2(\cdot)$ the dimensional constraints on B, and $\psi_3(\cdot)$ the dimensional constraints on C.

IV. A GEOMETRIC ALGORITHM FOR DATUM ESTABLISHMENT

From the previous sections, we see that datum establishment amounts to minimizing a linear function of the form

$$f(g, t) = t$$

subject to nonlinear constraints

$$d(g, y_i) + e_i(t) \le 0, \qquad i = 1, \ldots, n \tag{15}$$

where

$$d(g, y_i) = \langle g^{-1} y_i - x_i, n_i \rangle \tag{16}$$

$e_i(t)$ a linear function of t, $g \in Q$ and $Q = \{g_0 e^{\hat{m}} \,|\, \hat{m} \in \mathcal{M}\}$ is the configuration space of the underlying datum feature.

In general, the solution of a constrained minimization problem of the form

$$\min\{\psi(q) \,|\, \phi(q) \leq 0\}$$

where $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ is a C^1-function and $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ a set of nonlinear constraints, can be obtained by solving a sequence of linear programming (LP) problems with properly chosen initial conditions. To derive the corresponding LP problem, let $q^k \in \mathbb{R}^n$ be the initial condition satisfying the constraints and consider

$$q^{k+1} = q^k + \tilde{q} \tag{17}$$

where $\tilde{q} \in \mathbb{R}^n$ is a perturbation term. Computing the Tailor series expansion of $\psi(\cdot)$ and $\phi(\cdot)$ at q^k and retaining the first-order term yield

$$\psi(q^{k+1}) \approx \psi(q^k) + \langle d\psi(q^k), \tilde{q} \rangle$$

and

$$\phi(q^{k+1}) \approx \phi(q^k) + D\phi(q^k) \cdot \tilde{q}$$

where $d\psi(q^k)$ and $D\phi(q^k)$ are, respectively, the differential of ψ and the Jacobian of ϕ at q^k. The solution of the LP problem

$$\min\{\langle d\psi(q^k), \tilde{q} \rangle \,|\, \phi(q^k) + D\phi(q^k) \cdot \tilde{q} \leq 0\}$$

in (17) ensures that the constraints are satisfied while the function is minimized.

To apply the above method to establishment of DRF, we need to take into account that the underlying configuration space has the form $Q = \{g_0 e^{\hat{m}}\}$. In view of Section II, we consider perturbations of the form

$$(g^{k+1}, t^{k+1}) = (g^k e^{\hat{m}}, t^k + \tilde{t}) \tag{18}$$

where $\hat{m} \in \mathcal{M}$.

Choose a basis $(\hat{\eta}_1, \ldots, \hat{\eta}_r)$ of $\mathcal{M}$ and write

$$\hat{m} = \sum_{i=1}^{r} \hat{\eta}_i m_i$$

for some $m = (m_1, \ldots, m_r) \in \mathbb{R}^r$. Retaining the first order term of the Tailor series of $e^{\hat{m}}$ at I yields

$$e^{\hat{m}} \approx I + \hat{m}. \tag{19}$$

Substituting (19) into (16) and (15) yields the linearized constraints

$$\left\langle (I - \hat{m}) g_0^{-1} y_i - x_i, n_i \right\rangle + e_i(t_k) + e_i(\tilde{t}_k) \leq 0, \qquad i = 1, \ldots, n.$$

The linearized objective function is

$$\tilde{f}(m, \tilde{t}) = \tilde{t}.$$

Thus, the corresponding linear programming problem becomes

$$\min_{(m,\tilde{t}) \in \mathbb{R}^{r+1}} \left\{ \tilde{t} \,\middle|\, \left\langle (I - \hat{m}) g_0^{-1} y_i - x_i, n_i \right\rangle + e_i(t_k) + e_i(\tilde{t}_k) \leq 0 \right\}. \tag{20}$$

The preceding discussions are summarized into the following algorithm for establishment of DRF.

Algorithm 1: (Datum Establishment Algorithm)
Input: Measurement data set $Y = \{y_i\}_{i=1}^n, y_i \in \mathbb{R}^3$;
Output: (a) Established coordinate system $g^* \in Q$;
(b) Position or orientation tolerance values.
Step 0: (a) Set $k = 0$;
(b) Initialize g_0;
Set $t^0 = \max_i d(y_i, g^0)$;
(c) Solve for $x_i^0, i = 1, \ldots, n$;
(d) Compute $f^0 = t^0$;
Step 1: (a) Solve the LP problem (20) to obtain $(m, \tilde{t})$;
(b) Update (g^{k+1}, t^{k+1}) according to (18);
(c) Solve for $x_i^{k+1}, i = 1, \ldots, n$;
(d) Compute f^{k+1};
(e) If $(1 - f^{k+1}/f^k) > \epsilon$, then set $k = k + 1$ and return to Step 1(a); Else exit and report results.

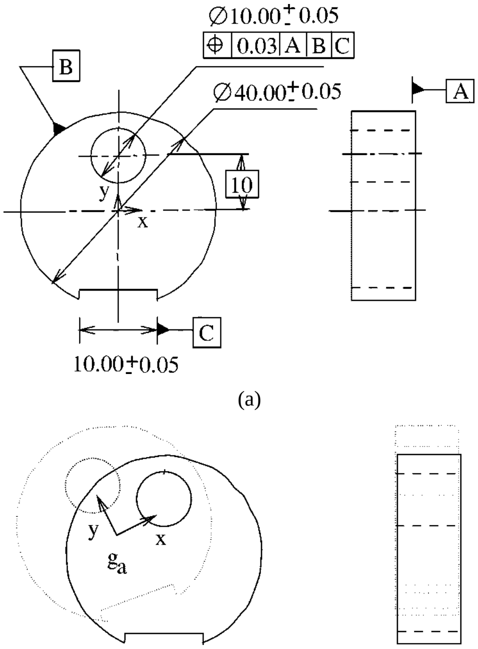
Remark 1: In general, datum callouts at MMC/LMC lose the constraints on datum features. The established coordinate system can "play" in certain region. Since the coordinate system created by RFS datums is one of the candidate coordinate systems, it is suggested that the solution of the RFS datums be used as the initial condition for the MMC/LMC problem. If the tolerance value with respect to this coordinate system conforms to the designed tolerance, the solution of RFS datums is regarded as the established coordinate system, otherwise, the formulation for MMC/LMC is applied.

Implementation details of the algorithm can be found in [6] and [5].

V. SIMULATION RESULTS

The algorithm proposed for establishing DRFs has been incorporated into a software package, called the Geometrical Tolerancing Package (GTPack). GTPack can be run on any UNIX based environment. We choose a SunSparc 4 workstation as the experimental platform. Solutions of linear programming problems are obtained using NAG mathematical package [9]. In this section, we apply the algorithm to three examples with and without MMC/LMC callouts to show the efficiency and simplicity of our approach. The last example uses real measurement data from a part machined by a machine center.

The measurement points in the first two examples are simulated by first choosing a set of points on all relevant features in a nominal configuration, then applying a known Euclidean transformation to these points and finally adding random noise to the transformed points. For computational convenience, we



(a)

(b)

Fig. 4. (a) Establishment DRF. (b) Coordinate system created by A.

set the number of measurement points on each feature surface to be the same, say 40. The initial condition is simply chosen to be the identity element of $\mathrm{SE}(3)$. The algorithm will finally recover the applied transformation.

Example 4 (DRF Without Datum Callout in MMC/LMC): Consider a mechanical drawing as shown in Fig. 4(a), which is a revised version of an example in ANSI Y14.5M.

The DRF for measuring the position tolerance of the hole is constructed from the primary datum plane A, the secondary datum cylinder B, and the tertiary datum width feature C. Datums B and C are called out at RFS condition. The nominal reference coordinate frame is established as shown in the figure. Thus, the Lie algebra $\mathcal{G}_A$ of G_A is spanned by

$$\mathcal{G}_A = \mathrm{span}\{\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_6\}$$

and a complementary space $\mathcal{M}_A$ is given by

$$\mathcal{M}_A = \mathrm{span}\{\hat{\xi}_3, \hat{\xi}_4, \hat{\xi}_5\}.$$

The configuration space of A is represented as

$$Q_A = \left\{ e^{(\lambda_1\hat{\xi}_3 + \lambda_2\hat{\xi}_4 + \lambda_3\hat{\xi}_5)} G_A \,\middle|\, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \right\}.$$

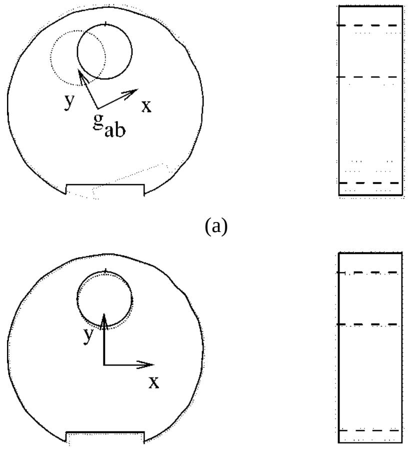
Solving Problem (3) on Q_A we obtain a transformation $g_a \in \mathrm{SE}(3)/G_A$.

Fig. 4(b) shows that with g_a, feature A is aligned with its nominal plane. Next, we analyze the configuration space of the secondary datum cylinder. The Lie algebra $\mathcal{G}_B$ of G_B is given by

$$\mathcal{G}_B = \mathrm{span}\{\hat{\xi}_3, \hat{\xi}_6\}.$$

Thus, the Lie algebra $\mathcal{G}_{AB}$ of G_{AB} is given by

$$\mathcal{G}_{AB} = \mathcal{G}_A \cap \mathcal{G}_B = \mathrm{span}\{\hat{\xi}_6\}.$$



(a)

(b)

Fig. 5. (a) Coordinate system by A-B. (b) Coordinate system by A-B-C.

Therefore, $\mathcal{G}_{AB}$ is a proper subset of $\mathcal{G}_A$, indicating the validity of the datum set A-B. We have

$$\mathcal{M}_B = \mathrm{span}\{\hat{\xi}_1, \hat{\xi}_2\}.$$

The configuration space of the secondary datum cylinder can be written as

$$Q_B = \left\{ g_a e^{(\lambda_1\hat{\xi}_1 + \lambda_2\hat{\xi}_2)} G_{AB} \,\middle|\, \lambda_1, \lambda_2 \in \mathbb{R} \right\}.$$

Solving (7) on Q_B yields the transformation g_{ab}. Fig. 5(a) shows the DRF with g_{ab} determined. Note that addition of B aligns B with its nominal feature while leaving A unchanged.

It is not difficult to see that the Lie algebra $\mathcal{G}_C$ of G_C is given by

$$\mathcal{G}_C = \mathrm{span}\{\hat{\xi}_2, \hat{\xi}_3, \hat{\xi}_4\}$$

and the Lie algebra $\mathcal{G}_{ABC}$ of A-B-C is

$$\mathcal{G}_{ABC} = \mathcal{G}_{AB} \cap \mathcal{G}_C = 0.$$

Thus, the addition of C fully constrains the free transformations left by A-B. The callout of datum C is valid. The configuration space of C is given by

$$Q_C = \{ g_{ab} e^{\lambda_1 \hat{\xi}_6} \mid \lambda_1 \in \mathbb{R} \}.$$

Fig. 5(b) shows the final DRF established by A-B-C, in which all datum features are sequentially aligned.

Table I shows the sequential results of the algorithm. The first row displays the applied transformation. The second row shows the DRF created by A, the third column of the rotation matrix gives the normal direction of the planar feature. The discrepancy is due to the random noise in the measurement data. The third row shows the DRF created by A-B. It is seen that the translational component of the transformation is recovered in this step. The fourth row shows the DRF created by A-B-C. The second and third columns display the computation iteration and computation time, respectively. The last row shows the bounds of the hole size with respect to the established DRF. Clearly, the hole is out of its MMC size. The workpiece should be rejected.

Example 5 (DRF with Datum Callout in MMC/LMC): We use the same mechanical drawing as in Example 1, except for

TABLE I
SIMULATION RESULTS WITHOUT MMC/LMC DATUM CALLOUT

	g_{WM}	Iter.	C. T.(s)
Applied transform	$R_0 = \begin{bmatrix} 0.5771 & -0.0.7070 & -0.4060 \\ 0.5771 & 0.7070 & -0.4060 \\ 0.5771 & 0.0 & 0.8120 \end{bmatrix}$ $q_0 = [6.0\ 10.0\ 4.0]^T$	-	-
-A-	$R_1 = \begin{bmatrix} 0.9083 & -0.0917 & -0.4082 \\ -0.0917 & 0.9083 & -0.4082 \\ 0.4081 & 0.4082 & 0.8166 \end{bmatrix}$ $q_1 = [1.2898\ 14.2555\ 3.7618]^T$	4	0.35
-A-B-	$R_2 = \begin{bmatrix} 0.9083 & -0.0917 & -0.4082 \\ -0.0917 & 0.9083 & -0.4082 \\ 0.4081 & 0.4082 & 0.8166 \end{bmatrix}$ $q_2 = [6.0078\ 9.9982\ 3.9921]^T$	5	0.25
-A-B-C-	$R_3 = \begin{bmatrix} 0.5774 & -0.7071 & -0.4082 \\ 0.5774 & 0.7071 & -0.4082 \\ 0.5772 & 0.0000 & 0.8166 \end{bmatrix}$ $q_3 = [6.0078\ 9.9982\ 3.9921]^T$	6	0.1
Tolerance	Sise=9.9454		

the secondary datum B being called out at MMC. Thus, the DRF created from the primary datum plane A can be obtained by the method of the previous example. The DRF created by A-B-C is obtained by minimizing

$$f(g, F) = -t$$

subject to the constraints

$$d(g, y_i) \geq t, \qquad i = 1, \ldots, n \tag{21}$$

$$d(g, b_j) \leq r_{\rm mmc}, \qquad j = 1, \ldots, m \tag{22}$$

$$s_{\rm mmc} \leq d(g, c_l) \leq s_{\rm lmc}, \qquad l = 1, \ldots, p \tag{23}$$

where

1) $Q = \{g_a e^{(m_1\hat{\xi}_1 + m_2\hat{\xi}_2 + m_3\hat{\xi}_6)} \,|\, m_1, m_2, m_3 \in \mathbb{R}\}$, and $g \in Q$;
2) t is the radius of the inscribed cylinder of the hole;
3) inequalities (21) assure that points on the hole lie within the position tolerance zone. $d(g, y_i)$ is the distance from the transformed point $g^{-1}y_i$ to the axis of the hole, $t_{\rm mmc}$ is the MMC size of the hole;
4) inequalities (22) assure that points on B lie within its MMC size $r_{\rm mmc}$;
5) inequalities (23) assure that feature satisfies its size requirements. $s_{\rm lmc}$ and $s_{\rm mmc}$ are the LMC and MMC size of C, respectively.

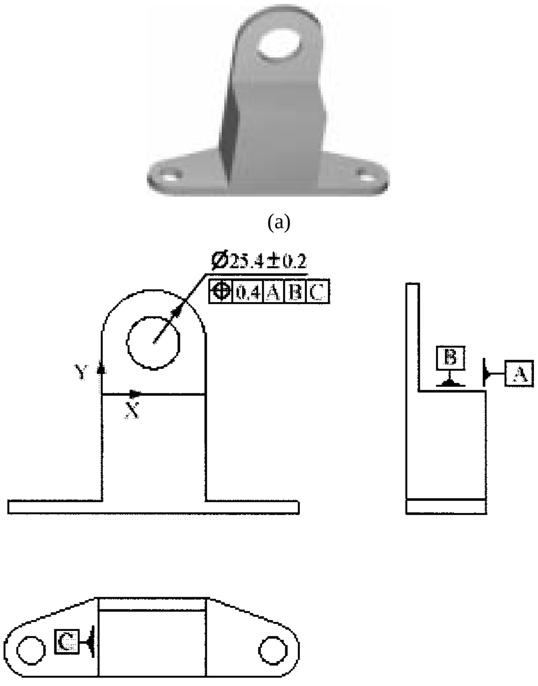
Table II shows the tolerance value and the established DRF. Since the secondary datum B is called out in MMC, the established coordinate system can "play" in certain region. Choice of the DRF intend to make the size of the hole larger than that of MMC. The second row shows the computed DRF. The last row displays the size and the position tolerance value of the hole. It is seen that both of them are in conformance to the requirements. The workpiece is then accepted.

Example 6 (DRF Using Measurement Data from a Real Part): Consider a part model shown in Fig. 6(a). The part is machined using a three-axis machine center. All measurement data were obtained using a touch-trigger probe. Again, 40 points were probed from each surface of the part.

As shown in Fig. 6(b), the datum reference system for locating the hole position is to be constructed from the primary datum plane A, the secondary datum plane B and the tertiary datum plane C. With the nominal reference coordinate shown as in the figure, the Lie algebra $\mathcal{G}_A$, $\mathcal{G}_B$, and $\mathcal{G}_C$ giving the respective datum feature

TABLE II
SIMULATION RESULTS WITH MMC/LMC DATUM CALLOUT

	g_{WM}	Iter.	C. T.(s)
-A-B-C- without MMC	$R_3 = \begin{bmatrix} 0.5774 & -0.7071 & -0.4082 \\ 0.5774 & 0.7071 & -0.4082 \\ 0.5772 & 0.0000 & 0.8166 \end{bmatrix}$ $q_3 = [6.0078\ 9.9982\ 3.9921]^T$	6	0.1
-A-B-C- with MMC	$R_3 = \begin{bmatrix} 0.5774 & -0.7071 & -0.4082 \\ 0.5774 & 0.7071 & -0.4082 \\ 0.5772 & 0.0000 & 0.8166 \end{bmatrix}$ $q_3 = [6.0109\ 10.0451\ 3.9921]^T$	3	0.2
Tolerance	Size=9.9528	PosTol=0.0062	



(a)

(b)

Fig. 6. (a) Part model. (b) Establishment DRF.

$$\mathcal{G}_A = \text{span}\{\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_6\}$$
$$\mathcal{G}_B = \text{span}\{\hat{\xi}_1, \hat{\xi}_3, \hat{\xi}_5\}$$
$$\mathcal{G}_C = \text{span}\{\hat{\xi}_2, \hat{\xi}_3, \hat{\xi}_4\}.$$

Note that

$$\mathcal{G}_{ABC} = \mathcal{G}_A \cap \mathcal{G}_B \cap \mathcal{G}_C = 0.$$

Thus, these three planes define a valid coordinate system (of course!).

Table III shows the results from the computation. The first three rows show the DRF created by A, A-B, and A-B-C respectively. The last row shows the position tolerance of the hole with respect to the established DRF. From the results in the table, we can see that the tolerance value of 0.3429 is within the tolerance bound and the part is accepted.

TABLE III
CALCULATION RESULTS FOR REAL PART

	$^{g}W\,M$	Iter.	C.T.(s)
-A-	$R_1 = \begin{bmatrix} 0.9286 & -0.0010 & 0.3710 \\ -0.0010 & 1.0000 & 0.0053 \\ -0.3710 & -0.0053 & 0.9286 \end{bmatrix}$ $q_1 = [5.4232 \ -16.6646 \ -6.7567]^T$	3	0.3
-A-B-	$R_2 = \begin{bmatrix} -0.0037 & -0.9286 & 0.3710 \\ 1.0000 & -0.0019 & 0.0053 \\ -0.0043 & 0.3710 & 0.9286 \end{bmatrix}$ $q_2 = [-7.8030 \ -2.5295 \ -1.5537]^T$	4	0.35
-A-B-C-	$R_3 = \begin{bmatrix} -0.0037 & -0.9286 & 0.3710 \\ 1.0000 & -0.0019 & 0.0053 \\ -0.0043 & 0.3710 & 0.9286 \end{bmatrix}$ $q_3 = [-7.6390 \ -46.8300 \ -1.3654]^T$	1	0.1
Tolerance	Size=25.6961		PosTol=0.3429

VI. CONCLUSION

In this paper, we developed a geometric theory for establishment of DRFs. By identifying the configuration space of a datum feature with a homogeneous space $\mathrm{SE}(3)/G_0$, where G_0 is the symmetry subgroup of the feature, we showed that datum development from datum features could be formulated as a minimization problem in $\mathrm{SE}(3)/G_0$. Furthermore, the problem of establishing coordinate systems from combinations of datum features can be formulated as a series of optimization problems. This formulation is not only consistent with the ANSI/Y14.5.1M standard but also suitable for CMM implementation. Utilizing properties of the homogeneous spaces we derived a simple and unifying algorithm for computation of DRFs. Simulation results showed several unique features of the algorithm as follows: a) consistency with the Y14.5.1M standard; b) computational efficiency; and c) simplicity of implementation.

A direct application of the main results contained here is for verification of position and orientation tolerances. We expect that it could also be used for systematically defining and analyzing assembly-dimension relations.

ACKNOWLEDGMENT

The authors would like to thank the unanimous reviewers for several useful suggestions.

REFERENCES

[1] W. Boothby, *An Introduction to Differentiable Manifolds and Riemannian Geometry*. New York: Academic, 1975.

[2] F. Etesami, "Position tolerance verification using simulated gaging," *Int. J. Robot. Res.*, vol. 10, no. 4, pp. 358–370, 1991.

[3] K. C. Fan and M. C. Leu, "Intelligent planning of CAD-directed inspection for coordinate measuring machines," *Comput. Integr. Manuf. Syst.*, vol. 11, no. 1–2, 1998.

[4] Q. Ge, B. Chen, P. Smith, and C. H. Menq, "Tolerance specification and comparative analysis for computer-integrated dimensional inspection," *Int. J. Prod. Res.*, vol. 30, no. 9, pp. 2173–2197, 1992.

[5] G. B. Gou, Y. X. Chu, and Z. X. Li, "A geomtric approach to establishment of datum reference frames," in *IEEE Int. Conf. Robotics and Automation*, 1999, pp. 1436–1441.

[6] J. B. Gou, "Theory and Algorithms for Coordinate Metrology," Ph.D. Thesis, Hong Kong Univ. Sci. and Technol., 1999.

[7] J. B. Gou, Y. X. Chu, and Z. X. Li, "On the symmetric localization problem," *IEEE Trans. Robot. Automat.*, vol. 14, pp. 533–540, Aug. 1998.

[8] *Mathematical Definition of Dimensioning and Tolerancing Principles, ANSI Standard Y14.5.1M*, 1994.

[9] *NAG Library Manual*, Tech. Rep., NAG Ltd., 1990.

[10] P. M. Martino, "Analysis of complex geometric tolerances in linear programming," in *Proc. 1992 ASME Computers in Engineering Conf.*, 1992, pp. 351–361.

[11] R. Murray, Z. X. Li, and S. Sastry, *A Mathematical Introduction to Robotic Manipulation*. Boca Raton, FL: CRC, 1994.

[12] W. Murray and M. Overton, "A projected Lagrangian algorithm for nonlinear minimax optimization," *SIAM J. Scientif. Statist. Computing*, vol. 1, no. 3, 1980.

[13] S. Nigam, J. Guilford, and J. Turner, "Derivation of generalized datum reference frames for geometric tolerance analysis," in *Proc. 19th Annu. ASME Design Automation Conf.*, 1993, pp. 159–166.

[14] R. Sodhi and J. Turner, "Relative positiong of variational part models for design analysis," *Computer-Aided Des.*, vol. 26, no. 6, 1994.

[15] J. Turner, S. Subramaniam, and S. Gupta, "Constraint representation and reduction in assembly modeling and analysis," *IEEE Trans. Robot. Automat.*, vol. 8, 1992.

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